Math 250
Practice Exam 3A

1. Let $f(x,y) = x^3 e^{y^2}$. Then
   (a). $f_{xy}(x,y) =$ (b). $f_{yy}(x,y) =$

   Solution: $f_x = 3x^2 e^{y^2}$, $f_y = 2yx^3 e^{y^2}$. And so,
   (a). $f_{xy} = 6yx^2 e^{y^2}$ (b). $f_{yy} = 2x^3 e^{y^2} + 4y^2 x^3 e^{y^2} = 2x^3 e^{y^2}\left(1+2y^2\right)$.

2. Evaluate the Iterated Integral: $\int_0^{\pi/6}\int_0^{\cos x} \sin^2 x \; dy\,dx$. Note: $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

   Solution: $\int_0^{\pi/6}\int_0^{\cos x} \sin^2 x \; dy\,dx = \int_0^{\pi/6} \cos x \sin^3 x \; dx = \frac{1}{3}\sin^3 x \Big|_0^{\pi/6} = \frac{1}{24}$.

3. Find the three critical points for $f(x,y) = x^2 y - 2xy + 2y^2 - 15y$, and for each one determine, *with justification*, if it is a local maximum, a local minimum, or a saddle point – show *clearly* how you got the critical points.
   Hint: The critical points have integer coordinates.

   Solution: $f_x = 2xy - 2y,\ f_y = x^2 - 2x + 4y - 15$. Solving $f_y = 0$ gives
   $y(x-1) = 0 \Rightarrow y = 0$ or $x = 1$. So, substituting these values into $f_y = 0$ yields
   $y = 0: x^2 - 2x - 15 = 0 \Leftrightarrow (x-5)(x+3) = 0 \Leftrightarrow x = 5$ or $x = -3$
   $x = 1: 4y - 16 = 0 \Leftrightarrow y = 4$.

   So the critical points are $(5,\ 0),\ (-3,\ 0),\ (1,\ 4)$.
   We have $f_{xx} = 2y,\ f_{yy} = 4,\ f_{xy} = 2x - 2$. So, in general, $D(x,y) = 8y - \left(2x-2\right)^2$.
   At (5, 0) and (–3, 0), $D < 0$ and so we have a saddle point at each of these.

   At (1, 4) we get $D = 32 > 0$ and so since $f_{xx}(1,4) = 8 > 0$, we get a local minimum here.

4. Find the point on the plane $3x + y + 2z = 28$ that is closest to the origin.
   Solution: We want to find the point on the plane $3x + y + 2z = 28$ at which $f(x,y,z) = x^2 + y^2 + z^2$ will be a minimum. Using Lagrange Multipliers,
   $3 = 2\lambda x,\ 1 = 2\lambda y,\ 2 = 2\lambda z \Rightarrow x = \frac{3}{2\lambda},\ y = \frac{1}{2\lambda},\ z = \frac{1}{\lambda}$. So, $x = 3y,\ z = 2y$.
   Hence, $9y + y + 4y = 28 \Rightarrow 14y = 28 \Rightarrow y = 2$. So the point must be $(6,\ 2,\ 4)$.

5. Reverse the order of integration. Show your work! In particular *sketch the relevant regions in each part*.

(a). $\int_0^2 \int_{1+y^2}^5 f(x,y)\, dxdy$.

Solution: $\int_0^2 \int_{1+y^2}^5 f(x,y)\, dxdy = \int_1^5 \int_0^{\sqrt{x-1}} f(x,y)\, dydx$

(b). $\int_0^3 \int_{y/3}^{4-y} f(x,y)\, dxdy$.

Solution: $\int_0^3 \int_{y/3}^{4-y} f(x,y)\, dxdy = \int_0^1 \int_0^{3x} f(x,y)\, dydx + \int_1^4 \int_0^{4-x} f(x,y)\, dydx$

6. Let $f(x,y) = 2x^2 + y^2 - 8x + 1$ be defined on the disk $x^2 + y^2 \le 25$. Find the global maximum and minimum values of $f(x,y)$. Explain your work carefully — determine *all* relevant critical points and show how you obtained them. Show your work in detail and *do not use Lagrange Multipliers.* Max__91___ Min_–7___

Solution: There is exactly one critical point of the form $\nabla f = 0$, namely (2, 0). We note that the value of *f* at this point is $f(2,0) = -7$.

The other critical points occur on the boundary of the disk, i.e., on the circle $x^2 + y^2 = 25$. On the circle we have $y^2 = 25 - x^2$ and so our function becomes $h(x) = f(x,y) = 2x^2 + (25 - x^2) - 8x + 1 = x^2 - 8x + 26, \ -5 \le x \le 5$.

Thus, since $h'(x) = 2x - 8 = 0 \Leftrightarrow x = 4$, we get the critical points on the circle corresponding to $x = 4,\ x = -5,\ x = 5$. Thus, since $h(4) = 10,\ h(-5) = 91,\ h(5) = 11$, we get that the minimum value of *f* is –7 and the maximum value so 91.

7. Find the length of the curve $f(x) = \frac{1}{3}\left(x^2 + 2\right)^{3/2}$ for $x = 0$ to $x = 1$.

Solution:

$$\int_0^1 \sqrt{1 + f'(x)^2}\ dx = \int_0^1 \sqrt{1 + f'(x)^2}\ dx =$$

$$\int_0^1 \sqrt{1 + \left(x\sqrt{x^2+2}\right)^2}\ dx = \int_0^1 \sqrt{1 + x^4 + 2x^2}\ dx = \int_0^1 \sqrt{\left[x^2+1\right]^2}\ dx =$$

$$= \int_0^1 x^2 + 1\ dx = \frac{1}{3}x^3 + x \Big|_0^1 = \frac{4}{3}.$$

8. $\int_0^1 \int_y^1 e^{x^2} \; dxdy$.

Solution:

$$\int_0^1 \int_y^1 e^{x^2} \; dxdy = \int_0^1 \int_0^x e^{x^2} \; dydx = \int_0^1 xe^{x^2} = \frac{1}{2}e^{x^2}\Big|_0^1 = \frac{e-1}{2}.$$

9. What is the minimum value of $f(x,y,z) = x + 2y + 4z$ subject to the constraint $xyz = 8$. (You may assume that the function has a minimum.).

Solution: $1 = \lambda yz \;\; 2 = \lambda xz, \;\; 4 = \lambda xy$. So, $x = 2y, \;\; x = 4z$ and so

$$8 = xyz = x\left(\frac{x}{2}\right)\left(\frac{x}{4}\right) \Rightarrow x^3 = 64 \Rightarrow x = 4.$$

Thus, $y = 2, \;\; z = 1$ and the only critical point is (4, 2, 1).
The minimum value of f then is $f(4,2,1) = 4 + 4 + 4 = 12$

10. Let D be the solid region bounded above by the paraboloid $z = 4 - x^2 - y^2$ and below by the xy-plane. Find the volume V of D.