Math 544
Review Problems for Exam 2

1. Let \( W \) be the set of all those polynomials \( p(t) \) in \( P_3 \) with \( p(3) = 0 \).
   Is \( W \) a subspace of \( P_3 \)?
   **Hint:** Check the three required properties and look at problem 3.
   **Answer:** Yes it is.
   **Solution:** Clearly 0 (which in this context means the polynomial that is the constant function 0) is in \( W \). If \( p(t) \) and \( q(t) \) are polynomials in \( W \), then
   \[ [p + q](3) = p(3) + q(3) = 0 + 0 = 0, \]
   and so \( p(t) + q(t) \) is in \( W \). Thus \( W \) is closed under the addition of vectors.
   Finally, if \( p(t) \) is in \( W \) and \( r \) is any real number, then \( rp(3) = r \times 0 = 0 \), so \( rp(t) \) is in \( W \) and hence \( W \) is closed under scalar products.

2. Let \( W \) be the set of all those polynomials \( p(t) \) in \( P_3 \) with \( p(0) = 3 \).
   Is \( W \) a subspace of \( P_3 \)?
   **Hint:** Consider the three required properties to be a subspace.
   **Answer:** No it is not.
   **Solution:** No, \( W \) does not contain the 0 vector.

3. Let \( W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : c = ab \right\} \subseteq R^3 \). Is \( W \) a subspace of \( R^3 \)?
   
   Justify your answer.
   **Solution:** No. \( W \) contains the zero vector, but is not closed under scalar multiplication.
   
   As an example to verify this consider \( \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \in W \), but \( 2 \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix} \notin W \).

4. Recall that a function \( f \) is even if \( f(x) = f(-x) \). Let \( W = \left\{ p(t) \in P_3 : p(t) \text{ is even} \right\} \subseteq P_3 \).
   Is \( W \) a subspace? If so, find a basis for \( W \).
   **Solution:** Yes, \( W \) is a subspace. The constant function 0 is in \( W \).
   If \( p(t) \) and \( q(t) \) are elements of \( P_3 \), then
   \[ (p + q)(-t) = p(-t) + q(-t) = p(t) + q(t) = (p + q)(t) \]
   and so \( W \) is closed under vector addition. Similarly, if \( r \) is any real number and \( p \) is an element of \( W \), then
   \[ (rp)(-t) = r(p(-t)) = r(p(t)) = (rp)(t) \]
   and so \( W \) is closed under scalar multiplication.
5. Let \( W = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a + b - 2c + d = 0, \) and \( a - b = c - d \) \( \subseteq \mathbb{R}^4. \)

Is \( W \) is a subspace of \( \mathbb{R}^4? \) Justify your answer.

**Solution:** Yes. Of course it is possible to verify that \( W \) is a subspace by checking the usual three basic properties that a subspace must possess, but here we can save a lot of time and effort by simply noting that \( W \) is the null space of the matrix \( A = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \) This is because the defining conditions of \( W \) can be written as \( a + b - 2c + d = 0 \) and \( a - b - c + d = 0. \)

Alternatively, we could reduce the defining conditions of \( W \) to \( a = 3b - d, \) \( c = 2b \) and so we would see that \( W \) is the span of \( \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \) \( \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \)

6. (a) Let \( V \) be a subspace of \( \mathbb{R}^n, \) and let \( V^\perp = \{ \mathbf{u} : \mathbf{u} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{v} \text{ in } V \}. \)

Show that \( V^\perp \) is a subspace of \( \mathbb{R}^n. \)

\( V^\perp \) is called the **orthogonal complement** of \( V. \)

**Solution:** Clearly \( \mathbf{0} \cdot \mathbf{u} = 0 \) and so \( \mathbf{0} \) belongs to \( V^\perp. \) If \( \mathbf{u} \) and \( \mathbf{w} \) belong to \( V^\perp \) then for any \( \mathbf{v} \) in \( V, \) \( (\mathbf{u} + \mathbf{w}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} = 0 + 0 = 0 \) and hence \( V^\perp \) is closed under vector addition. Similarly, you can show that \( V^\perp \) is closed under scalar multiplication.

(b) Let \( \mathbf{u} = \begin{bmatrix} 2 \\ -4 \\ 6 \\ -8 \end{bmatrix} \) and let \( V^\perp = \{ \mathbf{v} \in \mathbb{R}^4 : \mathbf{v} \cdot \mathbf{u} = 0 \}. \) Find a basis for \( V^\perp. \)

**Solution:** Suppose that \( \mathbf{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \) belongs to \( V^\perp. \) Then, from \( \mathbf{v} \cdot \mathbf{u} = 0, \) we get \( 2a - 4b + 6c - 8d = 0 \Rightarrow a = 2b - 3c + 4d. \)
So, $v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2b - 3c + 4d \\ b \\ c \\ d \end{bmatrix} = b \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - c \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ so, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \text{ spans } V^\perp.

Checking that these vectors are independent* shows that they form a basis for $V^\perp$.
*You could use the approach in problem 8 below or arguing

directly, $a \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2a + 3b + 4c \\ 0 \\ 0 \end{bmatrix}$ \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a = 0, b = 0, c = 0.

7. Let $W$ be spanned by $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. Find a basis for $W$.

**Solution:** If these three vectors formed an independent set, then they would also form a basis for $W$. However, forming the matrix $A$, using the given vectors as columns, we get that $A = \begin{bmatrix} 1 & 2 & 1 \\ 9 & 8 & 5 \\ 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and so these columns are not independent*, but the first two columns of $A$ (the pivot columns) are independent and form a basis for $W$.
*Note, in fact, that $a_2 = 3a_1 + 2a_3$ (where $a_i$ denotes the $i^{th}$ column of $A$).

8. Let $W = \{ A \in M_3 : AB = BA \}$. Where $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Find a basis for $W$.

\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}

**Solution:** A basis is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. 
9. Let $W$ be the subspace of all those polynomials $p(t)$ in $P_3$ with $p(3) = 0$. Determine a basis for $W$.

Solution: $p(t)$ belongs to $W$ precisely when 3 is a root of $p(t)$, which is equivalent to $t - 3$ being a factor of $p(t)$. So the elements of $W$ are of the form $(t - 3)q(t)$ where $q(t)$ has degree at most 2. Hence, a typical element of $W$ looks like $(t - 3)(ct^2 + bt + a) = ct(t - 3) + bt(t - 3) + a(t - 3)$ for some $a, b$ and $c$.

Hence $W$ is spanned by $\{t - 3, t(t - 3), t^2(t - 3)\}$.

These vectors are also independent and hence form a basis for $W$.

10. Suppose that $T : P_3 \to P_4$ is a linear transformation and that $T(1) = t^2 + 3t + 1$, $T(2t + 1) = t^3$, $T(1 + t^2) = 4t - 3$.

What is the value of $T(3t^2 + 8t + 12)$?

Answer: $4t^3 + 5t^2 + 27t - 4$.

Hint: Express $3t^2 + 8t + 12$ as a linear combination of $1, 2t + 1, 1 + t^2$.

Solution: $3t^2 + 8t + 12 = 3(1 + t^2) + 4(2t + 1) + 5$.

So,

$T(3t^2 + 8t + 12) = T[3(1 + t^2) + 4(2t + 1) + 5] = 3T(1 + t^2) + 4T(2t + 1) + 5T(1) = 3(4t - 3) + 4(t^3) + 5(t^2 + 3t + 1) = 4t^3 + 5t^2 + 27t - 4$.

11. Find bases for the null space, the column space, and the row space of $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix}$.

Solution: $A = \begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

So, a basis for $\text{Col}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. A basis for $\text{Nul}(A)$ is $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$.

A basis for $\text{Row}(A)$ is $\{[1, 2, 0, 0], [0,0,1,2]\}$. 
12. Find a basis for the vector space \( V = \left\{ a + 2b + 3c + d : a, b, c, d \text{ are real numbers} \right\} \).

\[
\begin{bmatrix}
a + 2b + 3c \\
b + c \\
a + d \\
a + 3b + 4c
\end{bmatrix}
= \begin{bmatrix}
a + 2b + 3c \\
b + c \\
a + d \\
a + 3b + 4c
\end{bmatrix}
= \begin{bmatrix}
a \\
b \\
a \\
a + 3b + 4c
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
1 \\
1
\end{bmatrix}
+ \begin{bmatrix}
2 \\
1 \\
0 \\
3
\end{bmatrix}
+ \begin{bmatrix}
3 \\
1 \\
1 \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
.
\]

Thus \( V \) is the same as the column space of \( A = \begin{bmatrix}
1 & 2 & 3 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 3 & 4 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & 3 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} \).

Thus the first three columns of \( A \) (the pivot columns) form a basis for \( V \).

13. Show that \( v = \begin{bmatrix} 7 \\ 18 \\ 12 \end{bmatrix} \) is in the span of \( S = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix} \) by writing \( v \) as an explicit linear combination.

\[
\begin{bmatrix} 7 \\ 18 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.
\]

14. Suppose that the columns of the \( n \times n \) matrix \( A \) are linearly independent. Explain why the system \( Ax = b \) must have exactly one solution for each vector \( b \).

**Solution:** Since the columns are all independent, there are \( n \) pivot positions and hence all rows of \( A \) are pivot rows and so the columns of \( A \) must span \( \mathbb{R}^n \). Thus every vector \( b \) is a linear combination of the columns of \( A \) and every linear combination of the columns of \( A \) has the form \( Ax \) for some vector \( x \).

15. Suppose that \( A \) is a \( 3 \times 5 \) matrix. Explain why it is not possible for the nullity of \( A \) to be 1.

**Solution:** Since the nullity(\( A \)) + rank(\( A \)) = 5, if nullity(\( A \)) = 1, then rank(\( A \)) = 4. However, 4 = rank(\( A \)) = dim(\( \text{Row}(A) \)) \leq 3 since there are only three rows. This is clearly impossible.
16. Let $T : P_3 \to M_2$ be defined by $T \left( p(t) \right) = \begin{bmatrix} p(0) & p(1) \\ p(2) & p(3) \end{bmatrix}$. Show that $T$ is 1-1.

**Hint:** A non-zero polynomial of degree 3 can have at most 3 different roots.

**Solution:** If $p(t)$ is in the kernel of $T$, then $p(0) = 0$, $p(1) = 0$, $p(2) = 0$, $p(3) = 0$.
But every non-zero polynomial of degree at most three can have at most three roots.
Hence $p(t) = 0$. Thus $\ker(T) = \{0\}$, and so $T$ is 1-1.

17. Let $A$ be an $m \times n$ matrix. Explain why $\mathbf{b}$ belongs to $\text{Col}(A)$ precisely when the system $A\mathbf{x} = \mathbf{b}$ is consistent.

**Solution:** $A\mathbf{x} = \mathbf{b}$ is consistent means that $\mathbf{b}$ is a linear combination of the columns of $A$ which is the same as saying that $\mathbf{b}$ belongs to $\text{Col}(A)$.

18. Show that the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a \\ a \\ b \end{bmatrix}$ is not onto.

**Solution:** $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not a functional value since $T \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

19. Let $T : M_2 \to \mathbb{R}^2$ be such that $T \left( \mathbf{v}_1 \right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $T \left( \mathbf{v}_2 \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $T \left( \mathbf{v}_3 \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T \left( \mathbf{v}_4 \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, where

$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Determine a basis for $\ker(T)$.

**Solution:** First we note that $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 + (d - c)\mathbf{v}_4$.

So that $T \left( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = aT \left( \mathbf{v}_1 \right) + bT \left( \mathbf{v}_2 \right) + cT \left( \mathbf{v}_3 \right) + (d - c)T \left( \mathbf{v}_4 \right) = \begin{bmatrix} a + c + 2d - 2c \\ 2a + b - c + d \end{bmatrix}$. 


Hence, a matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) belongs to \( \ker(T) \)
when
\[
\begin{bmatrix}
a - c + 2d \\
2a + b - c + d
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff a = c - 2d, \quad b = -c + 3d.
\]

So a typical element of \( \ker(T) \) has the form
\[
\begin{bmatrix}
c - 2d & -c + 3d \\
c & d
\end{bmatrix}
= c \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix}.
\]

So a basis for \( \ker(T) \) is \( \left\{ \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} \right\} \), since these vectors are independent and span.

20. Suppose that \( V \) and \( U \) are vector spaces with \( \dim V = \dim U = n \).
Note that \( V \cong U \) means that \( V \) and \( U \) are isomorphic vector spaces.
(a). Show that \( V \cong U \).

Solution:
Let \( \{v_1, v_2, \ldots, v_n\} \) be a basis for \( V \) and let \( \{u_1, u_2, \ldots, u_n\} \) be a basis for \( U \).
Then it is straightforward to show that the linear transformation \( T : V \to U \) defined by \( T(v_i) = u_i \) is 1-1 and onto and hence an isomorphism.

(b). Show that \( M_2 \cong P_3 \cong R^4 \).

Solution:
Use part (a) after noting that all three of these vector spaces have dimension 4.

21. When is the scalar 0 an eigenvalue of an \( n \times n \) matrix \( A \)?

Solution: Whenever \( A \) is not invertible.

22. If \( B \) is row equivalent to \( A \) must \( A \) and \( B \) have the same eigenvalues?

Solution: No. Row operations can change determinants. Counter-examples are plentiful.

23. \( v \) is in the eigenspace of the eigenvalue \( \lambda \) if \( v \) belongs to the null space of \( A - \lambda I_n \).

Solution: The null space of \( A - \lambda I_n \).
24. The characteristic polynomial, for the matrix below is \( p(\lambda) = (\lambda - 1)^3(\lambda - 2) \).

So \( \lambda = 1 \) is an eigenvalue of multiplicity 3.

\[
A = \begin{bmatrix}
-4 & 1 & 1 & 1 \\
-16 & 3 & 4 & 4 \\
-7 & 2 & 2 & 1 \\
-11 & 1 & 3 & 4
\end{bmatrix}
\]

Also, \( A - I \sim \begin{bmatrix}
-5 & 1 & 1 & 1 \\
0 & 3 & -2 & -2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \).

Determine a basis for the eigenspace of the eigenvalue \( \lambda = 1 \).

**Solution:** A basis is \( S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} \right\} \).

**Problems Previously Assigned in the Text:**
Section 4.1 #1, 2, 3, 5, 6, 7, 9, 11-18, 20, 21.
Section 4.2 #1, 2, 3, 5, 6, 7, 9, 11, 24, 28, 31, 32.
Section 4.3 #4, 10, 13, 15, 19, 25, 26.
Section 4.5 #1, 3, 5, 6, 13, 15, 17, 21.
Section 4.6 #1, 3, 5, 7, 9, 13, 15, 19, 21, 23, 27, 28, 29.
Section 3.1 #1, 9.
section 3.2 #3.2 #15 – 20.
Section 5.1 #1, 2, 3, 5, 7, 9, 11, 13, 19, 21, 22, 23–27, 29, 30.
Section 5.2 #1, 3, 7, 9, 15.

**Very Important Terms**

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