Method of Variation of Parameters
MATH 242 E01
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Consider an open interval I and the following nonhomogeneous linear differential equation

$$\text{(NH)} \quad y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots p_1(x)y' + p_0(x)y = f(x),$$

where $f(x)$ and each $y_i(x)$ are continuous on I.

We outline below how to use the Method of Variation of Parameters to find a particular solution $y_p(x)$ on I to (NH), for the case $n = 3$. For the other cases in which $n \geq 2$, similar steps also work, and for the case in which $n = 2$, the steps can be simplified, as noted below. If the functions $p_i(x)$ are not all constants, or if $f(x)$ is not of one of the types required by the Method of Undetermined Coefficients, then the Method of Variation of Parameters (but not the Method of Undetermined Coefficients) may be used to solve (NH).

The case $n = 3$:

Step 1: Find linearly independent solutions $y_1(x)$, $y_2(x)$, and $y_3(x)$ on I to the associated homogeneous equation (AH) of (NH), and let $W = W(y_1, y_2, y_3)$ be their Wronskian. For each i, $i = 1, 2, 3$, let W_i be the determinant of the 3×3-matrix obtained by replacing the i^{th} column of the matrix in W by the column
$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Evaluate W, W_1, W_2, and W_3.

Step 2: For each i, $i = 1, 2, 3$, use integration to find a formula for a function $u_i(x)$ such that for all $x \in I$, $u_i'(x) = \dfrac{W_i \cdot f(x)}{W}$.

Step 3: According to the Method of Variation of Parameters (verified in our text for the case $n = 2$ by using Cramer's Rule), $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + u_3(x)y_3(x)$ is a particular solution on I to (NH). Find a formula for y_p. Then by the methods studied previously, it follows that a general solution on I to (NH) is $y(x) = y_c(x) + y_p(x)$.

The case $n = 2$:

Step 1: Find linearly independent solutions $y_1(x)$ and $y_2(x)$ on I to the associated homogeneous equation (AH) of (NH), and let $W = W(y_1, y_2)$ be their Wronskian. For each i, $i = 1, 2$, let W_i be the determinant of the 2×2-matrix obtained by replacing the i^{th} column of the matrix in W by the column
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Evaluate W, W_1, and W_2.

Step 2: For each i, $i = 1, 2$, use integration to find a formula for a function $u_i(x)$ such that for all $x \in I$, $u_i'(x) = \dfrac{W_i \cdot f(x)}{W}$, i.e., so that $u_1'(x) = \dfrac{-y_2(x) \cdot f(x)}{W}$, and $u_2'(x) = \dfrac{y_1(x) \cdot f(x)}{W}$. *The latter two formulas are easily committed to memory, and one may prefer to learn them rather than the ones in which W_1 and W_2 appear.*

Step 3: Find a formula for a particular solution $y_p(x)$ on I to (NH) by evaluating the expression $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$. Then a general solution on I to (NH) is $y(x) = y_c(x) + y_p(x)$.

Exercises: Solve $y'' + y = \csc x$, and solve $y''' + y' = \csc x$ on $I = (0, \pi)$.