Math 242, Section E01
Information on the set $\mathbb{C}$ of Complex Numbers
February 20, 2017

Definition: If A is a set and x is an element of A, one writes "$x \in A$."

Definition: By a *complex number* is meant an ordered pair (a, b) of real numbers. Furthermore, the complex number (a, b) is identified with, and sometimes referred to as, the point in the plane, $\mathbb{R} \times \mathbb{R}$, whose cartesian coordinates are (a, b).

Definition: By the *field* $\mathbb{C}$ *of complex numbers* is meant the system $(\mathbb{R} \times \mathbb{R}, \oplus, \odot)$ where addition $\oplus$ and multiplication $\odot$ are defined by the following formulas. For all $(a, b), (c, d) \in \mathbb{C}$, $(a, b) \oplus (c, d) = (a + c, b + d)$, and $(a, b) \odot (c, d) = (ac - bd, ad + bc)$.

For simplicity, it is customary to modify the notation for a complex number $(a, b) \in \mathbb{R} \times \mathbb{R}$ and the binary operations $\oplus$ and $\odot$ in the following way: since (a, b) can be uniquely represented in the form $a(1, 0) \oplus b(0, 1)$, where $r(u, v)$ is defined to mean (ru, rv), we reserve the letter "i" to denote the complex number $(0, 1)$, and we abbreviate the notation for $(1, 0)$ to simply the number "1" (since $(a, b) \odot (1, 0) = (a, b) = (1, 0) \odot (a, b)$ for all $(a, b) \in \mathbb{R} \times \mathbb{R}$). This permits the following notationally simplified definition of $\mathbb{C}$, which is the one we shall use from now on.

Definition: By the *field* $\mathbb{C}$ *of complex numbers* is meant the set of all expressions having the form $a+bi$, where $a, b \in \mathbb{R}$, endowed with the following operations of addition and multiplication (which are denoted with the same symbols used for addition and multiplication of real numbers): for all $a, b, c, d \in \mathbb{R}$,

$(a + bi) + (c + di) = (a + c) + (b + d)i$, and $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$.

With these operations on $\mathbb{C}$, $0 + 0i$, denoted 0, is its additive identity, and the additive inverse of $a + bi$ is $-a + (-b)i$, denoted $-a - bi$.

Its multiplicative identity is $1 + 0i$, denoted 1, and for $0 \neq a + bi \in \mathbb{C}$, the multiplicative inverse of $a + bi$, denoted $\dfrac{1}{a + bi}$, is easily shown to be $\dfrac{a}{a^2 + b^2} - \dfrac{b}{a^2 + b^2} \cdot i$.

Notice that using the above formula for multiplication, one obtains $(0 + i) \cdot (0 + i) = (0 - 1) + (0 + 0)i$, and hence $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, and so on. Some authors write the suggestive notation $i = \sqrt{-1}$, although the symbol "$\sqrt{-1}$" is considered meaningless or undefined when one is talking solely about $\mathbb{R}$.

Definition: For a complex number $z = a + bi$, where $a, b \in \mathbb{R}$, a is called the *real part of* z, b is called the *imaginary part of* z, and the number $\sqrt{a^2 + b^2}$ is called the *modulus of* z (or *absolute value of* z) and is denoted $|z|$.

Geometrically, we can think of the complex number $z = a + bi$, where $a, b \in \mathbb{R}$, as being represented by the point (a, b) in the xy-plane. Then the modulus of z provides the distance between the point z and the origin. In this context, the x-axis is called the *real axis*, the y-axis is the *imaginary axis*, and the xy-plane is the *complex plane*.

Definition: If $a, b \in \mathbb{R}$ and $0 \neq z = a + bi$ is a nonzero complex number, then we can also write $z = |z| \left(\dfrac{a}{\sqrt{a^2+b^2}} + \dfrac{b}{\sqrt{a^2+b^2}} \cdot i \right)$, and since the point $\left(\dfrac{a}{\sqrt{a^2+b^2}}, \dfrac{b}{\sqrt{a^2+b^2}} \right)$ is obviously on the unit circle in the plane, then there exists a real number θ such that $\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin\theta = \dfrac{b}{\sqrt{a^2+b^2}}$. Thus we may write $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$. The latter is called the *polar form of* z, and θ is called an *argument of* z. Hence, the rectangular coordinates of the point z in the plane are (a, b), and a set of polar coordinates for z are (r, θ). As is the case for polar coordinates of any point, note that the argument θ of z is not uniquely determined, since in the preceding representation for z, θ may be replaced by any real number having the form $\theta + 2n\pi$, where n is an integer. So, θ denotes the signed radian measure of any angle whose initial side is the positive x-axis and whose terminal side is the line segment from 0 to z.

We shall use the following straightforward consequence of the formula for multiplication in $\mathbb{C}$ and the formulas $\cos(A+B) = \cos A \cos B - \sin A \sin B$ and $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

Lemma: If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are complex numbers, then $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i(\sin(\theta_1 + \theta_2))$.

Thus, to multiply two complex numbers together, we multiply their absolute values and add their angles. To add two complex numbers z_1, z_2, we view each complex number as being the tip of a position vector emanating from the origin, and then, geometrically, $z_1 + z_2$ is the tip of the vector sum $z_1 + z_2$.

In particular, one obtains the next result.

DeMoivre's Theorem: If $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is a positive integer, then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

We shall use a corollary of DeMoivre's Theorem to find roots of nonzero complex numbers.

Definition: If n is a positive integer and z, t are complex numbers such that $t^n = z$, then t is said to be an n^{th} *root of* z.

Corollary: If $z = r(\cos\theta + i\sin\theta)$, where $z \neq 0$ and n is a positive integer, then for each nonnegative integer k, $z_k = r^{1/n}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$ is an n^{th} root of z. Furthermore, z has exactly n distinct roots in $\mathbb{C}$, namely, $\{z_0, z_1, z_2, \ldots, z_{n-1}\}$.

Exercise: Let n be a positive integer. Let k be a nonnegative integer and define z_k by $z_k = \cos(\frac{2\pi k}{n}) + i\sin(\frac{2\pi k}{n})$. Show that z_k is an n^{th} root of 1.

Examples we have already seen: -1 and 1 are the square roots of 1, $-i$ and i are the square roots of -1, and $-i$, i, -1 and 1 are the fourth roots of 1.

Exercise: Use the Corollary and the two equations $1 = 1 \cdot (\cos(0) + i\sin(0))$ and $-1 = 1 \cdot (\cos(\pi) + i\sin(\pi))$ to generate the above square roots of -1 and 1, and the fourth roots of 1.

Exercise: Find formulas for all 3^{rd} roots of 1 and all 6^{th} roots of 1. Hint: use the above appropriate definitions of z_k, as well as trigonometry formulas such as $\cos(\pi/3) = 1/2 = \sin(\pi/6)$ and $\sin(\pi/3) = \sqrt{3}/2 = \cos(\pi/6)$, and similar formulas for $\cos\theta$ and $\sin\theta$, where θ denotes various integer multiples of $\pi/6$.

Exercise: Find formulas for all 4^{th} roots of -1. Hint: use $-1 = \cos\pi = 1\cdot(\cos\pi + i\sin\pi)$ and the above appropriate definitions of z_k, as well as trigonometry formulas such as $\cos(\pi/4) = \sqrt{2}/2 = \sin(\pi/4)$ and similar formulas for $\cos\theta$ and $\sin\theta$, where θ denotes various integer multiples of $\pi/4$.

Exercise: Find formulas for all 4^{th} roots of -16. Hint: $-16 = 2^4(-1)$.

Using the above results about $\mathbb{C}$ to help find solutions to homogeneous linear differential equations with constant coefficients:

Definition: The complex valued exponential function, denoted e^z, is defined in a way that its value at any real number agrees with the value given in courses on calculus and real numbers, and so that it has nice algebraic properties such as $e^{r+s} = e^r \cdot e^s$. As a result, one can derive, or take as a definition, the Euler formula, $e^{i\theta} = \cos\theta + i\sin\theta$, so that every nonzero complex number $z = r(\cos\theta + i\sin\theta)$ can be expressed as $z = re^{i\theta}$ or as $z = e^{\ln r + i\theta}$, where $r > 0$ and θ are real numbers.

Previously, we have shown that for a real number r, e^{rx} is a solution to a second order homogeneous differential equation with real constant coefficients $(*)\quad ay'' + by' + cy = 0$ iff r is a root to the characteristic equation $(**)\quad ar^2 + br + c = 0$ of $(*)$, and we have shown that if $b^2 - 4ac > 0$, then two linearly independent solutions to $(*)$ are $e^{r_1 x}$ and $e^{r_2 x}$, where r_1 and r_2 are the two distinct (real) roots to $(**)$. We also learned that if $b^2 - 4ac < 0$ (referred to as Case III), then two linearly independent solutions to $(*)$ are the functions $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$, where $\alpha = -b/2a$ and $\beta = \sqrt{4ac - b^2}/2a$. Using the previous definition, one can obtain the following.

Theorem: Let $(*)$ and $(**)$ be as above, where $b^2 - 4ac < 0$. Then, in $\mathbb{C}$, for any real numbers α and β, $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ are roots of $(**)$ if and only if $e^{r_1 x} = e^{(\alpha+i\beta)x}$ and $e^{r_2 x} = e^{(\alpha - i\beta)x}$ are solutions to $(*)$.

Since $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$, and hence $e^{(\alpha\pm i\beta)x}$ are the functions $e^{\alpha x}(\cos\beta x + i\sin\beta x)$ and $e^{\alpha x}(\cos\beta x - i\sin\beta x)$, then the above theorem also leads us to the (real) linearly independent solutions to $(*)$ that we obtained in our class, namely, $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$, where $\alpha = -b/2a$ and $\beta = \sqrt{4ac - b^2}/2a$. One can see this by noting that linear combinations of the functions in the preceding theorem produce the real valued solutions given in class, since $1/2$ of $e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x}$ produces $e^{\alpha x}\cos\beta x$, and $1/2i$ times $e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}$ produces $e^{\alpha x}\sin\beta x$.

Exercise: Find four linearly independent, real valued solutions to the homogeneous linear differential equation $(*)\quad y^{(4)} + 16y = 0$. Hint: The characteristic equation of $(*)$ is $(**)\quad r^4 + 16 = 0$. For two of the 4^{th} roots $\alpha + i\beta$ and $-\alpha + i\beta$ of $(**)$, i.e., of the number -16, form the functions $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$, $e^{-\alpha x}\cos\beta x$, and $e^{-\alpha x}\sin\beta x$. (There is no need to use the other roots of $(**)$ since $\cos(-\beta x) = \cos\beta x$ and $\sin(-\beta x) = -\sin\beta x$.)