Some Open Problems in Contest Theory

Alexander Matros

Motivation

- Applications
- Math background
- Open Questions

Nitzan (1994)

Contests can be characterized by
three main factors:

- 1.The contest is a n–player game, $n \geq 2$.
- 2.The contest rent is indivisible in the sense that one winner takes all.
- 3.The players expend effort (resources) to increase their probability of winning the rent.

Contests: applications

1. Military conflict,
2. Elections,
3. R&D,
4. Rent-seeking,
5. Lotteries,

and so on

Plan

- How does Contest Theory start?
- Tullock's Questions
- Private Information
- Experiments

Tullock (1980)

"Efficient rent-seeking"

Efficient rent-seeking

- "On the Efficient Organization of Trials"

  two-party conflict

  Lottery: $1 per ticket

$$P_A = \frac{A}{A+B}$$

Efficient rent-seeking

- "…[the odds] could be..." (p.225)

$$P_A = \frac{A^r}{A^r + B^r}$$

- More people

$$P_A = \frac{A^r}{A^r + B^r,\ldots,n^r}$$

Tullock (1980)

There are n players in a contest. Each player has to solve the following maximization problem

$$\max_{x_i} \frac{x_i^r}{\sum_{j=1}^{n} x_j^r} V - x_i,$$

where r is the marginal return to lobbying outlays. Note that there is a discontinuity at $(0, ..., 0)$ for any rule which determines a winner in this case.

Efficient rent-seeking: Appendix

When I first began working on this paper, I discovered that the equations that would have to be solved were higher-order equations, and therefore simply assigned to my graduate assistant, William J. Hunter, the job of approximating the results by using a pocket calculator. He promptly discovered the rather astonishing regularity of column 1, which implied that it would not be all that difficult to solve the equations even if they were higher order. Before I had had time to do anything other than shudder vaguely about the problem, however, I went to lunch with my colleague, Nicolaus Tideman, told him the problem, and he solved it on a napkin. This gave us the equation for tables 1 and 2. Having discovered this simple algorithm, when we wanted to prepare tables 3 and 4, once again we asked Tideman, and he obliged with equal speed. The equations used are:

Efficient rent-seeking: Appendix

$$P_A = R\frac{N - 1}{N^2} \qquad \text{(Tables 1, 2)}$$

$$P_A = R\frac{b}{(b + 1)^2} \qquad \text{(Tables 3, 4)}$$

where

P_A = equilibrium investment,

R = exponent, or the determinant of steepness of the supply curve,

N = number of players, and

b = bias weight.

Efficient rent-seeking

TABLE 1. *Individual Investments (N-person, No Bias, with Exponent)*

EXPONENT	NUMBER OF PLAYERS 2	4		10		15
1/3	8.33	6.25		3.00		2.07
1/2	12.50	9.37		4.50	I	3.11
1	25.00	18.75		9.00		6.22
2	50.00	37.50		18.00		12.44
3	75.00	56.25		27.00		18.67
5	125.00	93.75		45.00	II	31.11
8	200.00	150.00		72.00		49.78
12	300.00	225.00	III	108.00		74.67

Efficient rent-seeking

TABLE 2. ***Sum of Investments (N-Person, No Bias, with Exponent)***

EXPONENT	NUMBER OF PLAYERS 2	4	10	15		LIMIT
1/3	16.66	25.00	30.00	31.05		33.30
1/2	25.00	37.40	45.00	46.65	I	50.00
1	50.00	75.00	90.00	93.30		100.00
2	100.00	150.00	180.00	186.60		200.00
3	150.00	225.00	270.00	280.05		300.00
5	250.00	375.00	450.00	466.65	II	500.00
8	400.00	600.00	720.00	746.70		800.00
12	600.00	900.00	1,080.00 III	1,120.05		1,200.00

Tullock (1980)

- Tullock (1989) noted regarding the overdissipation result . . . "*when I demonstrated that perfect calculation leads to decidedly odd results even in a competitive market with free entry, I astonished myself*".

- He went on to note that the original (1980) paper "*was rejected by the Journal of Political Economy on the argument that it could not possibly be true that a competitive market would reach these results*".

Tullock (1980)

<u>**Proposition.**</u>

$$x^{*} = \frac{n-1}{n^{2}} rV$$

is the equilibrium spending, if $r \leq \dfrac{n}{n-1}$

Solving the rent-seeking game for r > 2

- Baye, Kovenock, and Vries (1994)
  - 2 players
  - Discrete strategy choice
  - Nash equilibrium (in mixed strategies) exists!

  - Experimental evidence in Millner and Pratt (1989)

Solving the rent-seeking game for r > 2

- Open question:

Find an analytical solution for r > 2.

Rent-Seeking with Asymmetric Valuation

Nti (1999)

Problem

- Each player maximizes her expected payoff:

$$\pi_i(x_1, x_2) = V_i \frac{x_i^r}{x_1^r + x_2^r} - x_i$$

Expected Payoff/Profit

Valuation

Tullock's Probability of Success Function

Cost

Timeline

	Tullock (1980)	Hillman (1989)	Nti (1999)
$V_1 \ldots V_n$	1	varies	varies
r	varies	1	varies
n	varies	varies	2

Analytical solutions

- Tullock (1980): $$x_i^* = \frac{n-1}{n^2} rV$$
- Hillman (1989): $$x_i^* = S - \frac{1}{V_i} S^2, \quad S = \frac{(n-1)n}{n\sum_{i=1}^{n}\left(1/V_n\right)}$$
- Nti (1999): $$x_i^* = \frac{rV_i^{r+1}V_j^r}{(V_1^r + V_2^r)^2}$$

Problem

- Each player maximizes her expected payoff:

$$\pi_i(x_1, x_2) = V_i \frac{x_i^r}{x_1^r + x_2^r} - x_i$$

Expected Payoff/Profit

Valuation

Tullock's Probability of Success Function

Cost

Open question

- Generalize for $n > 2$

	Tullock (1980)	Hillman (1989)	Nti (1999)	Future ?
$V_1 \ldots V_n$	1	varies	varies	varies
r	varies	1	varies	varies
n	varies	varies	2	varies

Contests

- Known number of players
- Perfect Information

Contests with Private Values

Malueg and Yates (PC, 2004)

Motivation

- The authors study a rent-seeking contest in which the players' valuations of the prize are private information.
- In this two players contest, each player knows his own valuation of the prize but is uncertain of the other's.

1. Model with Private Information

- Players X and Y competing for a prize.
- Value of prizes: V_x and V_y.
- Players bid simultaneously values x and y.
- $\pi(x, y)$ denote the probability that X wins the contest when bids are x and y.

Model

$$U_x(x,y) = \left(\frac{x^r}{x^r + y^r} \right) Vx - x$$

$$U_y(x,y) = \left(\frac{y^r}{x^r + y^r} \right) Vy - y$$

$$\pi(x, y) = \begin{cases} \frac{x^r}{x^r + y^r} & \text{if } (x, y) \neq (0, 0) \\ \frac{1}{2} & \text{if } (x, y) = (0, 0), \end{cases}$$

Model

- Vx and Vy are modeled as random variables.
- The prior probability distribution of (Vx, Vy) is:

		v_Y	
		v_L	v_H
v_X	v_L	$\frac{1}{2}\sigma$	$\frac{1}{2}(1-\sigma)$
	v_H	$\frac{1}{2}(1-\sigma)$	$\frac{1}{2}\sigma$

Model

$$E[U_X | v_X, (y_L, y_H)] = E\left[\frac{x^r}{x^r + y^r} | v_X, (y_L, y_H)\right] v_X - x,$$

- We can differentiate it with respect to x to obtain the FOC describing X's optimal bid, conditional on his value:

$$\frac{x}{r\, v_X} = E\left[\frac{x^r y^r}{(x^r + y^r)^2} | v_X, (y_L, y_H)\right]$$

Bayesian Equilibrium

- <u>*Proposition . (Bayesian equilibrium)*</u>

  If a symmetric pure-strategy Bayesian equilibrium exists, then it is unique and is given by (bL, bH) such that

  $$bL = \kappa VL \quad \text{and} \quad bH = \kappa VH$$

  where

  $$\kappa \equiv \left(\frac{\sigma}{4} + \frac{1-\sigma}{(\rho^{-r/2} + \rho^{r/2})^2}\right) r$$

  and

  $$\rho \equiv v_L / v_H$$

Contests with a Stochastic Number of Players

Lim and Matros

Games and Economic Behavior, 2009

The Model

n potential risk neutral players

V – prize value

$p \in (0,1]$ contest's participation probability

$X_i \geq 0$ player i's expenditure

Two interpretations

1. Contests with a stochastic number of players

Myerson and Warneryd, “Population uncertainty in contests,” Economic Theory (2006)

Munster, “Contests with an unknown number of contestants,” Public Choice (2006)

Two interpretations

2. Contests with Private values

Malueg and Yates, “Rent seeking with Private values,”

Public Choice (2004)

Players' maximization problem

Player i solves the following problem

$$\max_{X_i}\left[\sum_{M\in\Psi^{N_i}} p^{|M|}(1-p)^{|N_i/M|}P_i(X_i;M)\right]V - X_i$$

$$P_i(X_i;M)=\begin{cases}\dfrac{X_i^r}{X_i^r+\sum_{j\in M}X_j^r}, & if\;\; X_i>0,\\ 0, & if\;\; X_i=0.\end{cases}$$

N

$\mathbf{N_i} = \mathbf{N}/\{i\}$ – set of player i's possible opponents

Ψ^{N_i} - set of all subsets of $\mathbf{N_i}$

Results: unique equilibrium

Theorem.

Suppose that $0 \le r \le \frac{n+1}{n}.$

Then there exists a unique symmetric pure-strategy equilibrium and it is given by

$$X^*(r, V, n, p) = rV\left(\sum_{i=1}^{n-1} C_i^{n-1} p^i (1-p)^{n-i-1} \frac{i}{(i+1)^2}\right),$$

where

$$C_i^{n-1} = \frac{(n-1)!}{i!\,(n-i-1)!}$$

Results: Tullock (1980)

Corollary 1.

If $0 \le r \le \frac{n}{n-1}$ then

$$X^*(r,V,n,1) = \frac{n-1}{n^2} rV.$$

Open Question

- *What if*
  - *More than 2 players*
  - *More than two values*

- Malueg and Yates (2004)
  - 2 players
  - Two values: $0 \leq V \leq W$

- Lim and Matros (2009)
  - n players
  - Two values: 0 and $V > 0$

Open Question

- *How do people play lotteries?*
- *How to model such behavior?*

Shogren and Baik, PC 1991

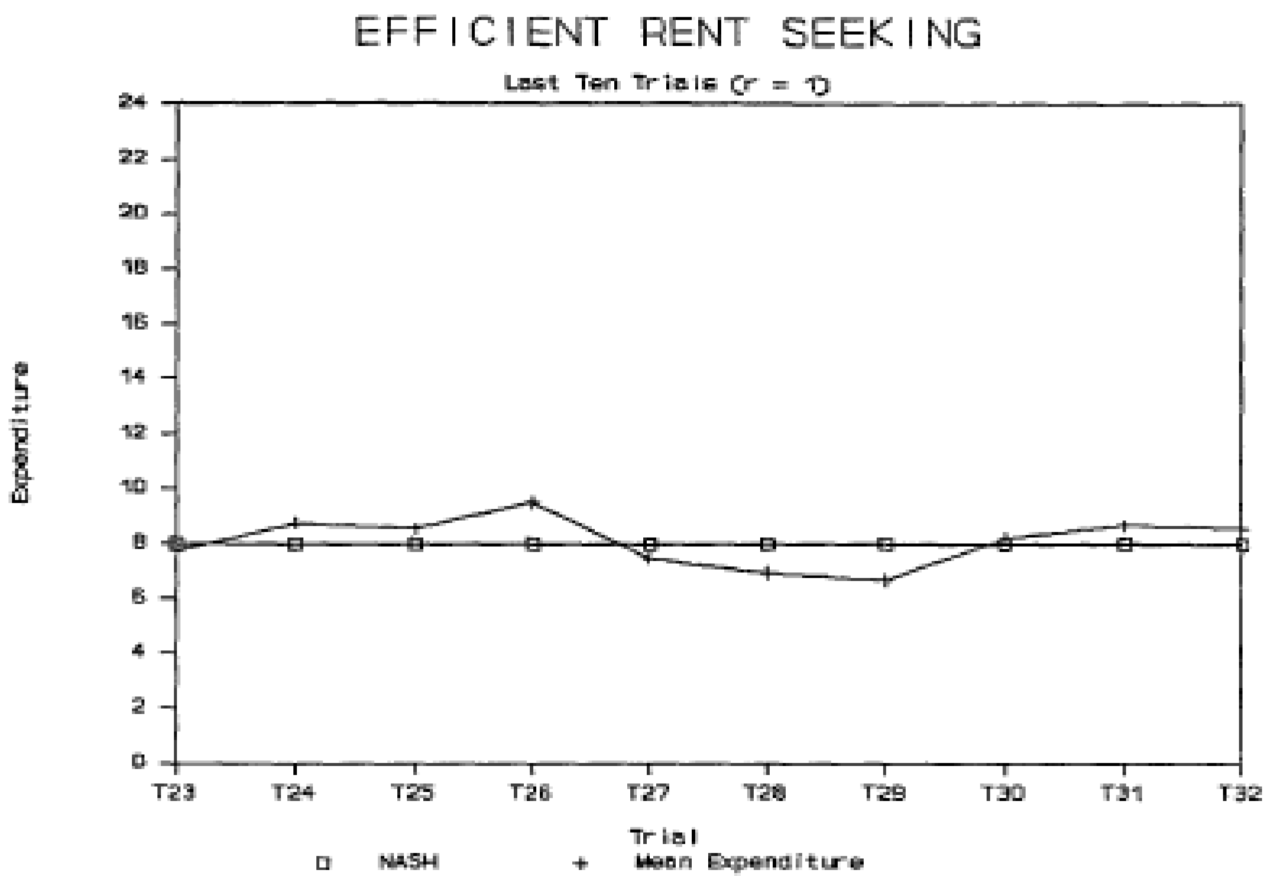


Figure 3. Mean rent-seeking expenditures (T23–T32).

Lotteries

- As of 2008, 43 States have State Lotteries
- 33% - 50% of USA population participates

Lotteries

Too many players buy *too many* tickets

Why?

Literature

- (A) Buy Hope?

Clotfelter and Cook (1989, 1990, 1993)

Literature

- (A) Buy Hope?

Clotfelter and Cook (1989, 1990, 1993)

- (B) Charity/Fund raising?

Morgan (2000), Morgan and Sefton (2000)

Literature

- (A) Buy Hope?

Clotfelter and Cook (1989, 1990, 1993)

- (B) Charity/Fund raising?

Morgan (2000), Morgan and Sefton (2000)

- **What if no (A) and no (B)?**

Theory

n risk neutral players

V – prize value

W – endowment

$x_i \geq 0$ player i's expenditure

Players' maximization problem

Player i solves the following problem

$$\max_{x_i} u_i\left(x_1,\ldots,x_i,\ldots,x_n\right) \quad (1)$$

$$u_i\left(x_1,\ldots,x_i,\ldots,x_n\right)=\begin{cases} w-x_i+\left(\dfrac{x_i}{\sum_{j=1}^{n} x_j}\right)V, & if\ x_i>0,\\ w, & if\ x_i=0.\end{cases}$$

Nash equilibrium

- Absolute performance

$$x_1^\star = \dots = x_N^\star = x^\star(N) = V \cdot \frac{N-1}{N^2}$$

- **Unique** Nash equilibrium!

Evolutionary Stable Strategies

- Relative performance (spiteful behavior)

$$x^{ESS} = \frac{V}{n}$$

Experimental Design

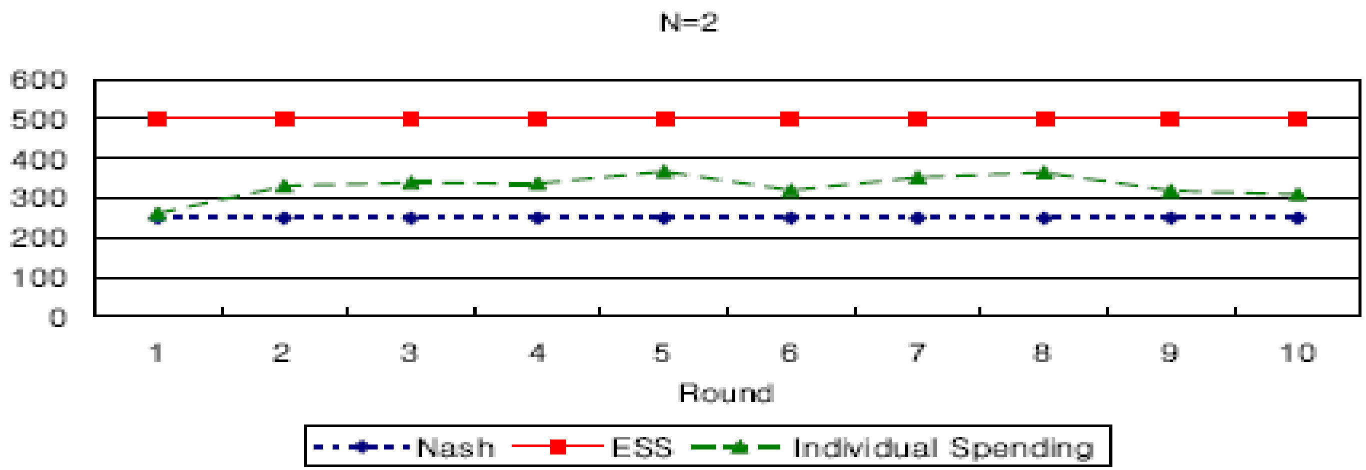
V = 1,000 tokens (= $10)
W = 1,200 tokens (= $12)
Quizzes
Expected payoff tables

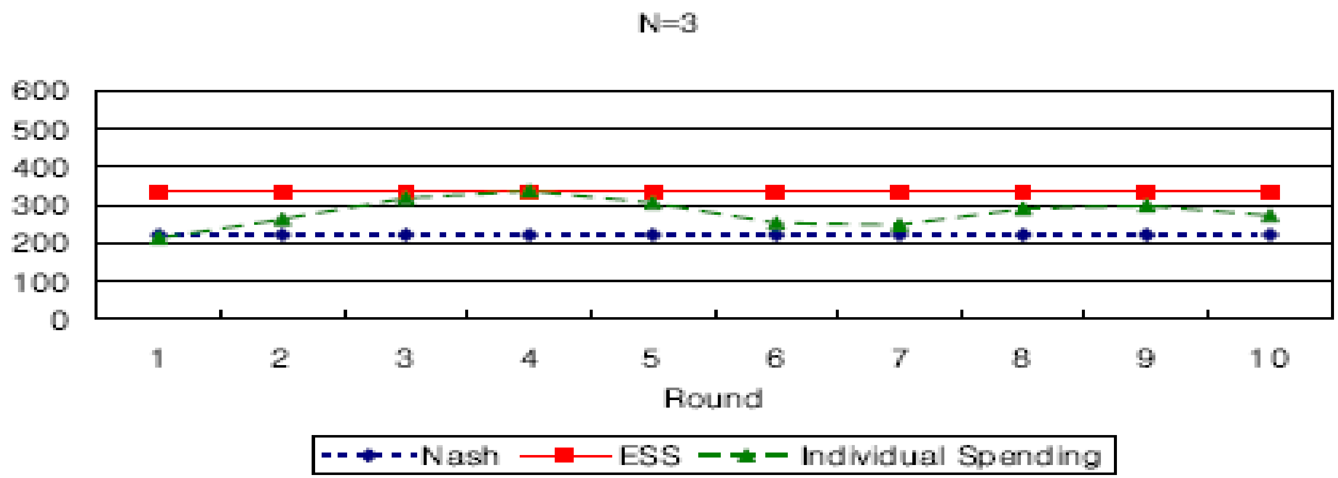
N = 2, 3, 4, 5, 9
3 sessions for each N
Pittsburgh Experimental Economics Laboratory
October 2007 – March 2008

Experimental Design

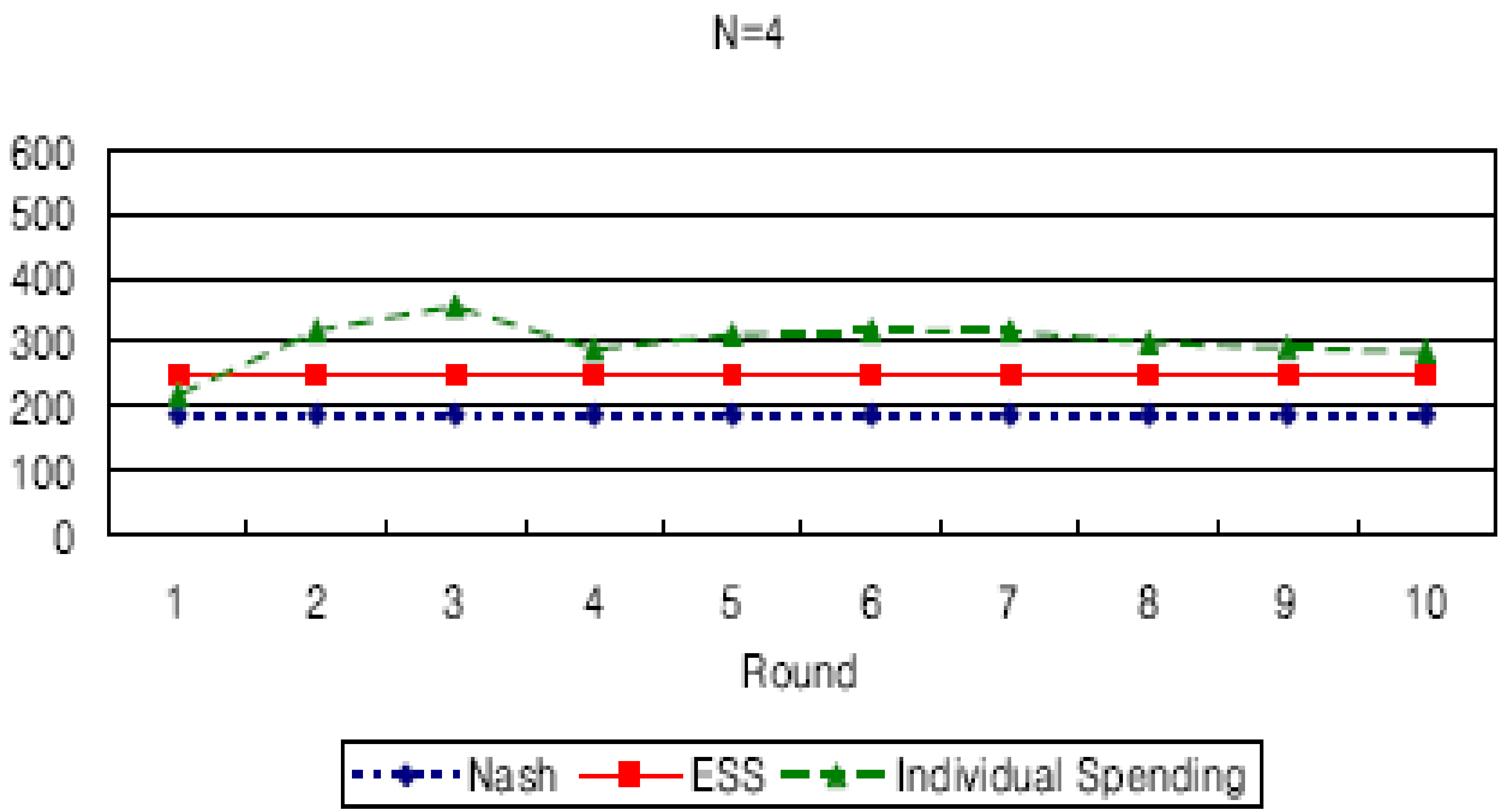
Expected Payoff	YOUR OPPONENT'S BID													
YOUR BID		0	100	200	300	400	500	600	700	800	900	1000	1100	1200
	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	100	900	400	233	150	100	67	43	25	11	0	−9	−17	−23
	200	800	467	300	200	133	86	50	22	0	−18	−33	−46	−57
	300	700	450	300	200	129	75	33	0	−27	−50	−69	−86	−100
	400	600	400	267	171	100	44	0	−36	−67	−92	−114	−133	−150
	500	500	333	214	125	56	0	−45	−83	−115	−143	−167	−188	−206
	600	400	257	150	67	0	−55	−100	−138	−171	−200	−225	−247	−267
	700	300	175	78	0	−64	−117	−162	−200	−233	−263	−288	−311	−332
	800	200	89	0	−73	−133	−185	−229	−267	−300	−329	−356	−379	−400
	900	100	0	−82	−150	−208	−257	−300	−338	−371	−400	−426	−450	−471
	1000	0	−91	−167	−231	−286	−333	−375	−412	−444	−474	−500	−524	−545
	1100	−100	−183	−254	−314	−367	−413	−453	−489	−521	−550	−576	−600	−622
	1200	−200	−277	−343	−400	−450	−494	−533	−568	−600	−629	−655	−678	−700

N = 2, 3: Nash and ESS





N = 4


N=4
600
500
400
300
200
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2
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4
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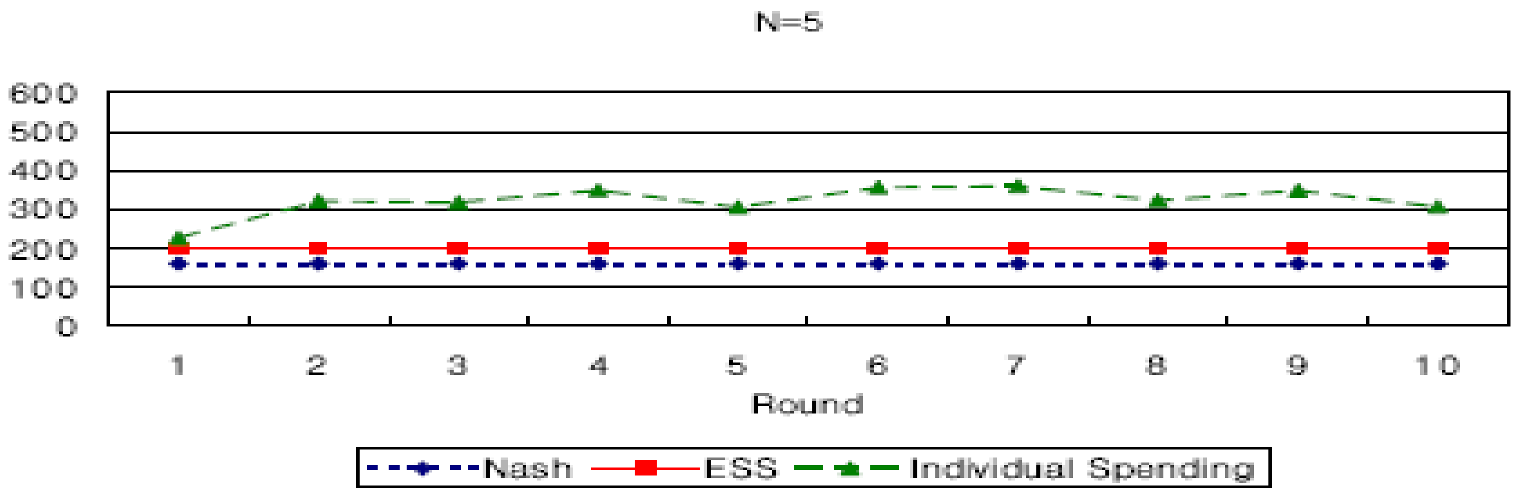
6
7
8
9
10
Round
Nash
ESS
Individual Spending

N = 5, 9


N=5
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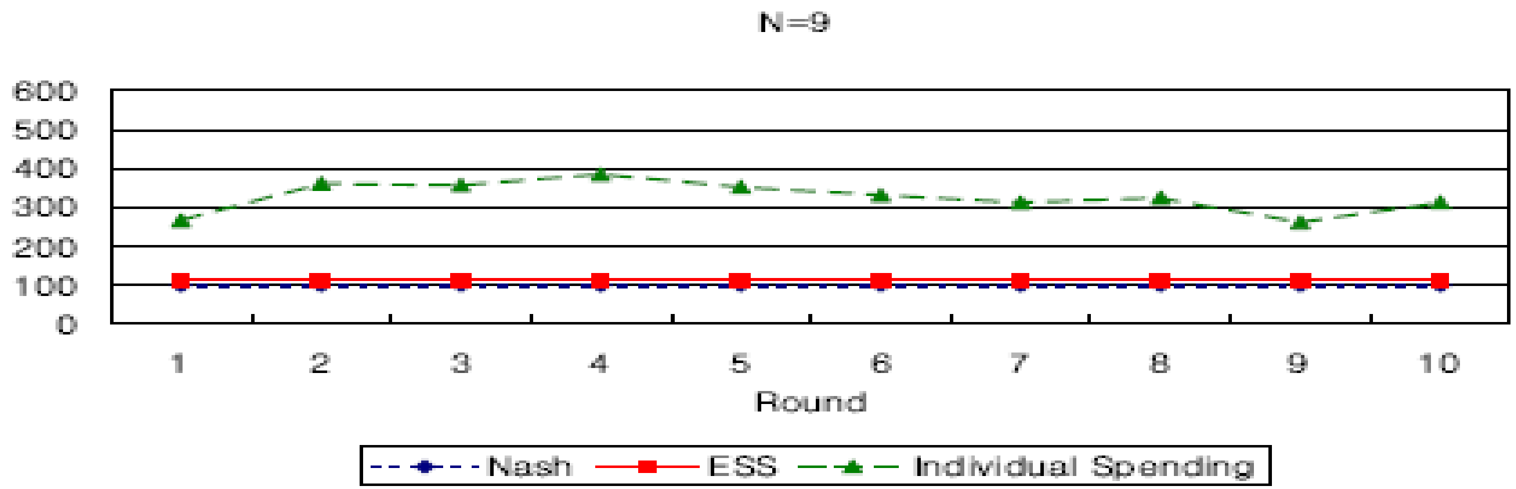
Round
Nash
ESS
Individual Spending


N=9
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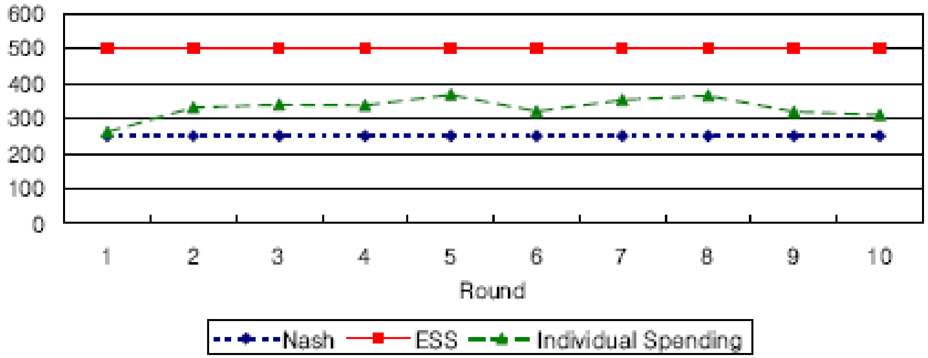
Round
Nash
ESS
Individual Spending

N=2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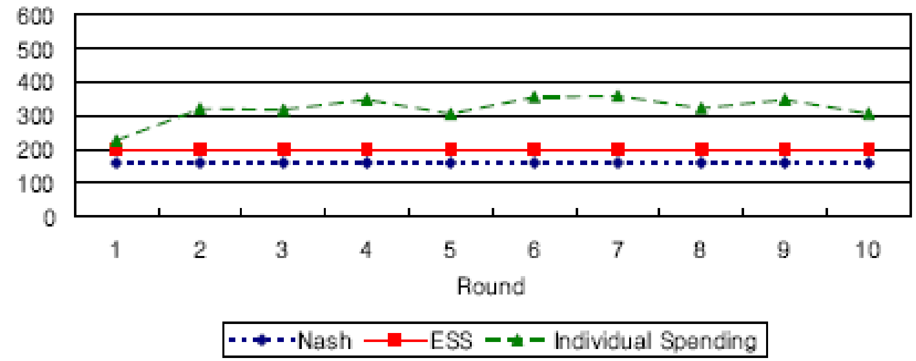
Round
Nash
ESS
Individual Spending

N=5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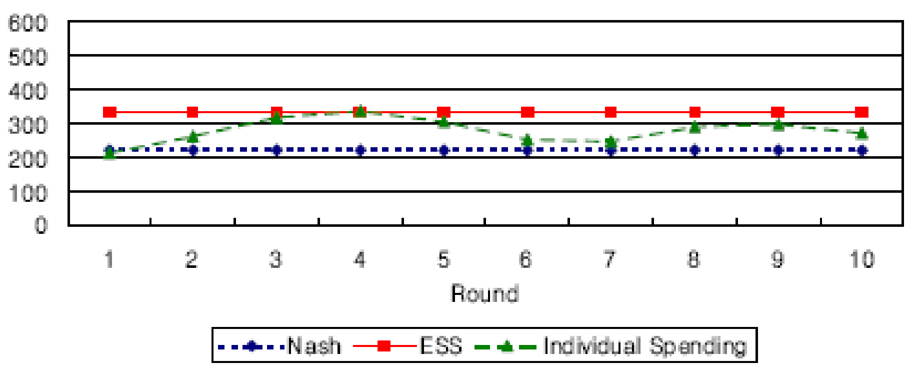
Round
Nash
ESS
Individual Spending

N=3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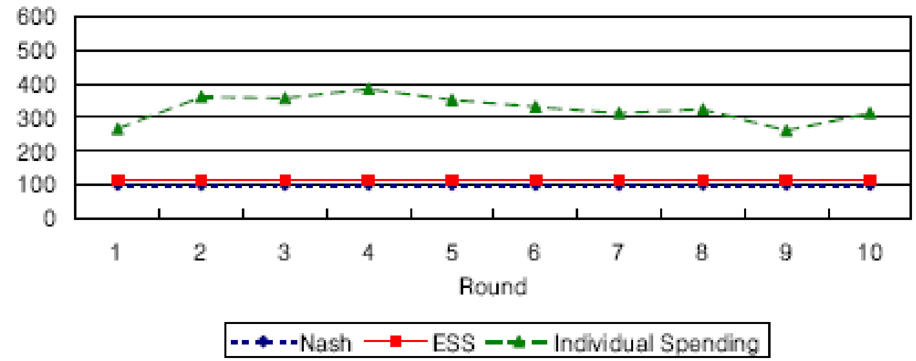
Round
Nash
ESS
Individual Spending

N=9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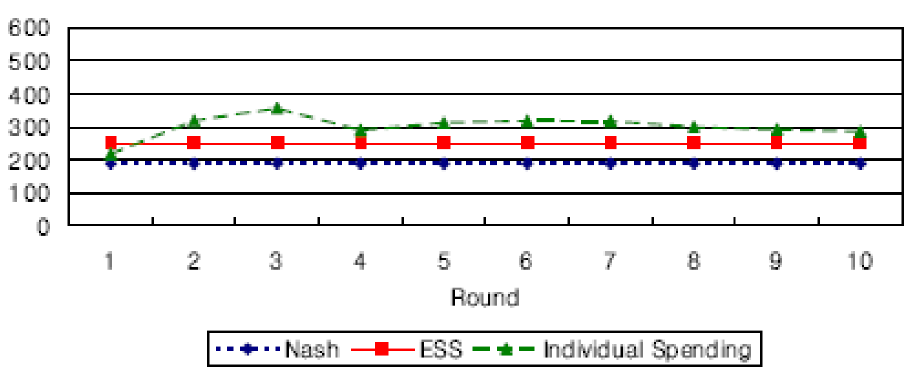
Round
Nash
ESS
Individual Spending

N=4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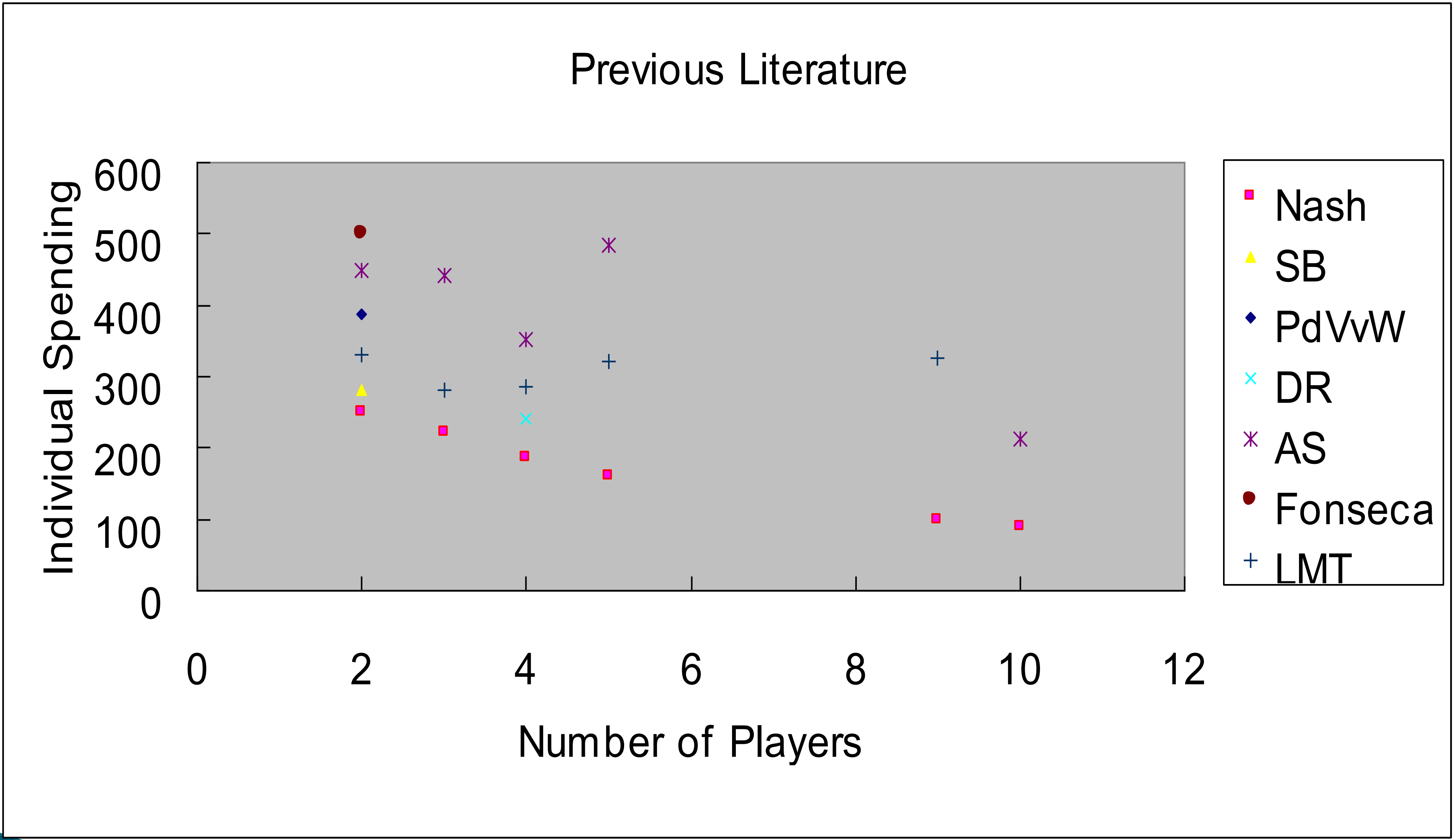
1 2 3 4 5 6 7 8 9 10
Round
Nash
ESS
Individual Spending



- MP: Millner and Pratt (1989)
- SP: Shogren and Baik (1991)
- PdVvW: Potters, de Vries, van Winden (1998)
- DR: Davis and Reilly (1998)
- AS: Anderson and Stafford(2003)
- Fonseca: Fonseca (2006)
- LMT: Our result

Author(s)	Year	N	Individual Spending (average)					Note
			Theoretical Prediction		Experiment (C)	% Difference		
			Nash (A)	ESS (B)		100*{(C)-(A)}/(A)	100*{(C)-(B)}/(B)	
Millner and Pratt	1989	2	250	500	280	12.0	-44.0	
Shogren and Baik	1991	2	250	500	253.4	1.4	-49.3	Expected-payoff Table
Potters, de Vries, van Winden	1998	2	250	500	388.5	55.4	-22.3	Expected-payoff Table
Davis and Reilly	1998	4	187.5	250	242.1	29.1	-3.2	
Anderson and Stafford	2003	2	250	500	448	79.2	-10.4	
		3	222	333.4	441.7	99.0	32.5	
		4	187.5	250	351.5	87.5	40.6	
		5	160	200	483.4	202.1	141.7	
		10	90	100	212.7	136.3	112.7	
Fonseca	2006	2	250	500	500.4	100.2	0.1	
Lim, Matros, and Turocy	–	2	250	500	330.6	32.2	-33.9	Expected-payoff Table
		3	222	333.4	280.1	26.2	-16.0	
		4	187.5	250	286.6	52.9	14.6	
		5	160	200	321.2	100.8	60.6	
		9	98.8	111.11	326.2	230.2	193.6	

* These values are normalized with V=1,000 to make a comparison with our results.

Open Question

- How to explain overspending in lotteries?
- How to model subjects' behavior?