1. Problems on Cesaro means:
   (a) Turn in (after all) the earlier assigned problem on Cesaro means of sequences: Convergence implies Cesaro summability.
   (b) Suppose that both \( \{a_n\}_n \) and \( \{b_n\}_n \) are Cesaro summable, then show that the sequence \( c_n := \alpha a_n + \beta b_n \) is also.
   (c) Determine the Cesaro means of the sequence \( \{(-1)^n - \frac{1}{n}\} \).

2. How can you tell if a norm actually arises from an inner product? Use a ‘bullet’ to show that the inner product in an inner product space over the real scalars can be computed from the norm by the formula:

\[
< f, g > = \frac{1}{2} \left( \|f\|^2 + \|g\|^2 - \|f - g\|^2 \right)
\]

Extra Credit: Derive a similar formula when the scalar field is \( \mathbb{C} \).

3. Consider the collection \( \phi_n(t) = \cos(nt) \) in \( L^2(-\pi, \pi) \). Show that this collection is orthogonal where the inner product is given by \( \int_{-\pi}^{\pi} f(t)g(t)dt \). What is the norm of \( \phi_n \)? (Hint: use the trig identity \( \cos(a + b) + \cos(a - b) = 2 \cos(a) \cos(b) \).)

4. A metric space is called separable if it has a countable dense subset. Let \( H \) be a Hilbert space with the natural metric (i.e. \( d(f, g) := \|f - g\|_H \)) and let \( \Phi := \{\phi_\alpha\}_\alpha \) be any orthonormal collection from \( H \).
   (a) Compute the distance between any two distinct members of \( \Phi \).
   (b) Prove that \( \Phi \) must be countable if \( H \) is separable.
   (c) Sketch the proof that if \( \Phi \) is countable and maximal in the partial ordering of set inclusion (existence by Zorn’s lemma), then \( H \) is separable.

5. Let \( X \) be a Banach space (i.e. a complete normed linear space). Prove that if \( M_n := \|f_n\|_X \) and the sequence \( \{\sum_{n=1}^{N} M_n\}_N \) is bounded in \( \mathcal{R} \), then the series \( \sum_{n=1}^{\infty} f_n \) converges in \( X \). (Recall that for the series converge, we just mean that the sequence of partial sums convergence as a sequence in \( X \).)