# ANALAYIS II

## Ratio and n-th Root Tests

### (using lim sup and lim inf)

**Defn.** If $\{a_n\}$ is a sequence of real numbers, then define the ***limit inferior*** and ***limit superior***, respectively, by

$$\liminf_{n\to\infty} a_n := \sup_k (\inf_{n \ge k} a_n)$$
$$\limsup_{n\to\infty} a_n := \inf_k (\sup_{n \ge k} a_n)$$

**Note.** If we define $\alpha_k := \inf_{n \ge k} a_n$, then it is clear that the $\{\alpha_n\}$ form a nondecreasing sequence and will converge in the extended sense. Similarly, $\beta_k := \sup_{n \ge k} a_n$ form a nonincreasing sequence and will converge in the extended sense to its infimum.

**Proposition.** Suppose $\alpha = \liminf_{n\to\infty} a_n$, then for each $\varepsilon > 0$, eventually $\alpha-\varepsilon < a_n$ and infinitely often $a_n < \alpha+\varepsilon$. Similarly, if $\limsup_{n\to\infty} a_n = \beta$, then for each $\varepsilon > 0$ eventually $a_n < \beta+\varepsilon$ and infinitely often $\beta < a_n + \varepsilon$.

> *Proof.* We prove the first statement and leave the other for the student. By the definition of the *limit inferior*, we see that if $\alpha_k := \inf_{n \ge k} a_n$, then for $\varepsilon > 0$ there is an n such that $\alpha-\varepsilon < \alpha_k \le \alpha$. The statement of the theorem follows directly from the definition of infimum applied to $\alpha_k$. □

**Corollary.** If $\liminf_{n\to\infty} a_n > a$, then eventually $a_n > a$. Similarly, if $\limsup_{n\to\infty} a_n < b$, then eventually $a_n < b$.

> *Proof.* For the first statement, let $\alpha := \liminf_{n\to\infty} a_n$, set $\varepsilon = \alpha-a$, and apply the previous proposition. For the second, set $\varepsilon = b-\beta$. □

**Corollary.** A sequence $\{a_n\}$ converges if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ . The common value is the limit of the sequence.

> *Proof.* Apply the previous proposition. □

**Theorem.** (**Ratio Test**) For a sequence of nonnegative numbers, define

$$R := \limsup_{n\to\infty} a_{n+1}/a_n$$
$$r := \liminf_{n\to\infty} a_{n+1}/a_n$$

then for the series $\Sigma_{n=1}^{\infty} a_n$

- $R < 1$ implies convergence,
- $r > 1$ implies divergence,
- $R = 1$, $r = 1$ the test is inconclusive.

*Proof.* To prove the first statement is true, we observe as shown above that $R < 1$ implies that eventually $a_{n+1}/a_n \leq T$ where T is strictly between R and 1. So there exists N such that $0 \leq a_{n+1} \leq T\, a_n$ for all $N \leq n$. By induction we then see that eventually (i.e. for $N \leq n$), we have $a_n \leq C\, T^n$ where $C: = a_N/T^N$. Applying the comparison test and the fact that $0 < T < 1$, we see that the series converges. On the other hand, if $r > 1$, then a similiar argument shows that eventually $a_n > C\, t^n > C$, where $r>t>1$ and $C = a_N/t^N$ for some N. Hence by the n-th term test, the series must diverge. The last statement of the theorem follows since $\Sigma_{n=1}^{\infty} 1/n$ diverges, $\Sigma_{n=1}^{\infty} 1/n^2$ converges, but $R = r = 1$ for both series. □

**Note.** The special limit $\lim_{n\to\infty} n^{1/n} = 1$ will be useful in what follows.

Details: By taking logarithms and using the continuity of the log function at x=0, we see that it will suffice to show that $\lim_{n\to\infty} \log(n)/n = 0$. This follows however by an application of L'Hospital's rule. □

**Theorem.** (**n-th Root Test**) For a sequence of nonnegative numbers, define

$$R := \limsup_{n\to\infty} (a_n)^{1/n}$$

then for the series $\Sigma_{n=1}^{\infty} a_n$

- $R < 1$ implies convergence,
- $R > 1$ implies divergence,
- $R = 1$ implies the test is inconclusive.

*Proof.* To prove the first statement is true, we observe as shown above that $R < 1$ implies that eventually $(a_n)^{1/n} \leq T$ where T is strictly between R and 1. So there exists N such that $0 \leq a_n \leq T^n$ for all $N \leq n$. Applying the comparison test and the fact that $0 < T < 1$, we see that the series converges. On the other hand, if $R > 1$, then infinitely often $a_n > ((R+1)/2)^n > 1$, so by the n-th term test, the series must diverge. The third part of the theorem follows since $\Sigma_{n=1}^{\infty} 1/n$ diverges, $\Sigma_{n=1}^{\infty} 1/n^2$ converges, but $\lim_{n\to\infty} n^{1/n} = 1$ and so $R = 1$ for both series. □

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