ANALYSIS II
Introduction to Series

Defn. Let X be a complete normed linear space. Suppose $\{x_n\}_{n=1}^{\infty}$ belongs to X, then the ***infinite series*** $\sum_{k=1}^{\infty} x_k$ is said to ***converge*** if the sequence of partial sums $s_n := \sum_{k=1}^{n} x_k$ converges. In this case, $\sum_{k=1}^{\infty} x_k := \lim_{n\to\infty} s_n$. In particular, this definition applies to the real and complex scalar fields.

Examples:

1. Let X = IR, then $\sum_{n=1}^{\infty} 1/(k^2 + k) = 1$.
   Details: Set $a_k = 1/(k^2 + k)$ and notice that $a_k = 1/k - 1/(k+1)$. Observe that the sum telescopes, $s_n = \sum_{k=1}^{n} 1/k - 1/(k+1) = 1 - 1/(n+1) \to 1$ as $n\to\infty$.
2. Let X be the complex numbers, then $\sum_{n=1}^{\infty} a z^n$ converges to $s = a/(1-z)$ if $|z| < 1$, and diverges otherwise.
   Details: As we have seen before, $|s_n-s| = |a z^n/(1-z)| \le C r^n$ for some constant ($C = |a|/(1-r)$) and $r = |z|$. We see by the theorem that follows (*n-th term test*) that the series diverges if $1 \le |z|$.
3. Let X = C[0,1/2] and let $f_n(x) = x^n$, then $\sum_{n=1}^{\infty} x^n$ converges in X to the function $f(x) = x/(1-x)$.

Proposition. Suppose that $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ converges in X and c is a scalar, then

1. $\sum_{k=1}^{\infty} c\, x_k = c \sum_{k=1}^{\infty} x_k$
2. $\sum_{k=1}^{\infty} (x_k + y_k) = \sum_{k=1}^{\infty} x_k + \sum_{k=1}^{\infty} y_k$

Proof. The proof follows immediately from the corresponding properties for sequences applied to the sequences of partial sums. □

Theorem. (**Cauchy Test**) In a complete normed linear space X, a series $\sum_{k=1}^{\infty} x_k$ converges if and only if for each $\varepsilon > 0$, there is a natural number N such that for $m > n \ge N$ $\| \sum_{k=n+1}^{m} x_k \|_X < \varepsilon$.

Proof. $\sum_{k=1}^{\infty} x_k$ converges if and only if the sequence of partial sums $\{s_n\}$ is Cauchy. The theorem follows since $s_m-s_n = \sum_{k=n+1}^{m} x_k$ if $m > n$. □

Corollary. (**n-th Term Test**) Suppose that $\sum_{k=1}^{\infty} x_k$ converges in X, then $\lim_{n\to\infty} x_n = 0$ in X.

> ***Proof***. Let n = m-1 in the Cauchy test. □

Corollary. (**Weierstrass M-Test**) Suppose $\{f_n\}$ is a sequence of continuous functions on [a,b] which satisfies $\|f_n\|_\infty \le M_n$ where $\sum_{n=1}^{\infty} M_n < \infty$, then the series $\sum_{n=1}^{\infty} f_n$ converges in C([a,b]) to some f; that is, there exists a continuous function f on [a,b] such that $\sum_{k=1}^{n} f_k$ converges uniformly to f as $n\to\infty$.

> ***Proof***. We apply the Cauchy test to S_n, the n-th partial sum for the series $\sum_{n=1}^{\infty} f_n$. Note that $\|S_m - S_n\| = \| \sum_{k=n+1}^{m} f_k \| \le \sum_{k=n+1}^{m} M_k$ by the triangle inequality. Applying the Cauchy criterion to the right hand side of this inequality, since $\sum_{k=1}^{\infty} M_k$ converges, then completes the proof. □

Note. In the previous theorem, the same proof gives a corresponding theorem for the space of continuous functions (with `sup norm') on Ω, where Ω is a compact metric space.

Theorem. (**Positive Term Test**) Suppose each a_n is nonnegative, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums is bounded. We then can write $\sum_{n=1}^{\infty} a_n = \sup_n s_n$ in the extended sense.

> ***Proof***. The sequence of partial sums with nonnegative terms is a monotone nondecreasing sequence which will converge to its least upper bound. □

Defn. A statement p(n) is said to be ***eventually*** true if there exists a natural number N such that p(n) is true for every n ≥ N. A statement p(n) is said to be true ***infinitely often*** if for each natural number N, there exists n ≥ N such that p(n) is true.

Corollary. Suppose that eventually the terms of the series are nonnegative, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums is bounded.

> ***Proof***. Eventually the sequence of partial sums are nondecreasing. □

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