Analayis II
Series Convergence Tests

Corollary. (**Comparison Test**) Suppose that eventually $0 \le a_n \le b_n$, then

- if $\Sigma_{n=1}^{\infty} b_n$ converges, so does $\Sigma_{n=1}^{\infty} a_n$.
- if $\Sigma_{n=1}^{\infty} a_n$ diverges, so does $\Sigma_{n=1}^{\infty} b_n$.

Proof. Let t_n be the n-th partial sums of $\Sigma_{n=1}^{\infty} b_n$ and s_n be the n-th partial sums of $\Sigma_{n=1}^{\infty} a_n$. Then eventually $|s_n - s_m| \le |t_n - t_m|$. □

Defn. A series $\Sigma_{n=1}^{\infty} x_n$ in a normed linear space X is said to ***converge absolutely*** if $\Sigma_{n=1}^{\infty} \|x_n\|_X$ converges. Of course, the real and complex number systems are special cases.

Theorem. (**Absolute Convergence Test**) If a series converges absolutely, then it converges.

Proof. Let s_n be the sequence of partial sums of $\{x_n\}$ and $\{t_n\}$ be that for $\{\|x_n\|\}$, then for $m > n$ the triangle inequality implies $\|s_n - s_m\| = \|\Sigma_{k=n+1}^{m} x_n\| \le \Sigma_{k=n+1}^{m} \|x_n\| = t_m - t_n$. □

Theorem. (**Limit Comparison Test**) Suppose we have two nonnegative sequences which satisfy $\lim_{n\to\infty} a_n/b_n = \alpha$ with $0 < \alpha < \infty$. Then $\Sigma_{n=1}^{\infty} a_n$ and $\Sigma_{n=1}^{\infty} b_n$ converge and diverge together.

Proof. Without loss of generality, we can assume that all terms are nonnegative. By the hypothesis we see that eventually $r\, a_n < b_n < R\, a_n$ where $r = \alpha/2$ and $R = 2\,\alpha$. □

Theorem. (**Alternating Series Test**) Suppose that a_n decreases to 0, then the series $\Sigma_{n=1}^{\infty} (-1)^{n+1} a_n$ converges (to s say). Furthermore, the error estimate holds:

$$\left| \sum_{k=1}^{n} (-1)^{k+1} a_k - s \right| \le a_{n+1}$$

Proof. Consider the sequence of partial sums $\{s_{2n}\}$, then these are monotone since $0 \le a_{2k-1} - a_{2k}$ and $s_{2n} =$

$s_{2n-2} + (a_{2n-1} - a_{2n})$. This sequence is also bounded since we can rewrite and estimate it as $s_{2n} = a_1 - a_{2n} - \Sigma_{k=1}^{n-1} (a_{2k} - a_{2k+1}) \le a_1$. Hence the sequence of s_{2n}'s converge. Since the odd terms of the sequence of partial sums satisfy $s_{2n+1} = s_{2n} + a_{2n+1}$ and $a_{2n+1} \to 0$, we see that the sequence s_n also converges to s. For the error estimate, we may estimate the difference of partial sums by

$$| s_n - s_{n+2k+1} | = | a_{n+1} - \Sigma_{j=1}^{k} (a_{n+2j} - a_{n+2j+1}) | \le |a_{n+1}|.$$

Since the absolute value function is continuous and $s_{n+2k+1} \to s$ as $k \to \infty$, the error estimate follows in the limit. □

Example. $\Sigma_{n=1}^{\infty} (-1)^{n+1}$ 1/n converges but does not converge absolutely.

Details: Convergence of the series follows directly from the alternating series test. To show the series is not absolutely convergent, consider the function f(x) = 1/x and the partition P = {1,2,...,n+1} of [1,n+1], then $\log(n) \le \int_1^{n+1} f(x)\,dx \le U(P,f) = \Sigma_{k=1}^{n} 1/k$. But we know that $\log(n) \to \infty$, which shows that the series $\Sigma_{n=1}^{\infty} 1/n$ does not converge.

Theorem. (**Integral Test**) Suppose that f is nonnegative and monotone decreasing on $[1,\infty)$, then $\Sigma_{n=1}^{\infty} f(n)$ converges if and only if $\lim_{n\to\infty} \int_1^n f(x)\,dx$ is finite.

Proof. Simply note that

$$\Sigma_{n=1}^{\infty} f(n) \sim \lim_{n\to\infty} \int_1^n f(x)\,dx.$$

by considering upper and lower Riemann sums. □

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