Theorem. Suppose that $\alpha_1$, $\alpha_2$ are non-decreasing, and that $f$, $g$ are Riemann-Stieltjes integrable with respect to both $\alpha_1$ and $\alpha_2$. If $c$ is a nonnegative real number, then

1. $\int_a^b c f \, d\alpha = c \int_a^b f \, d\alpha = \int_a^b f \, d(c \alpha)$
2. $\int_a^b f+g \, d\alpha = \int_a^b f \, d\alpha + \int_a^b g \, d\alpha$
3. $\int_a^b f \, d(\alpha+\beta) = \int_a^b f \, d\alpha + \int_a^b f \, d\beta$
4. $g \leq f$ implies $\int_a^b g \, d\alpha \leq \int_a^b f \, d\alpha$.
5. $|\int_a^b f \, d\alpha| \leq \int_a^b |f| \, d\alpha \leq ||f||_\infty (\alpha(b) - \alpha(a))$
6. $\int_a^b f \, d\alpha = f(s)$ if $f$ is continuous at $s$ between $a$ and $b$, and $\alpha(x) = I(x-s)$ where $I(x) = 1$ for positive $x$ and vanishes otherwise.
7. $\int_a^b f \, d\alpha = \sum_{n=1}^N c_n f(s_n)$ if $\alpha(x) = \sum_{n=1}^N c_n I(x-s_n)$, $f$ is continuous at each of the $s_n$'s (all of which are assumed to lie in the interval $(a,b)$), and the $c_n$'s are nonnegative.

Proofs:

Property 1: We observe that $c \geq 0$ implies $c \alpha$ is nondecreasing, $M_i(c \, f) = c \, M_i(f)$ and $m_i(c \, f) = c \, m_i(f)$. Hence $U(P; c \, f, \alpha) = c \, U(P; f, \alpha) = U(P; f, c \alpha)$. A similar statement holds for lower sums.

Property 2: We notice that $M_i(f+g) \leq M_i(f) + M_i(g)$ and $m_i(f) + m_i(g) \leq m_i(f+g)$. Hence,

$$L(P; f, \alpha) + L(P; g, \alpha) \leq L(P; f+g, \alpha) \leq U(P; f+g, \alpha) \leq U(P; f, \alpha) + U(P; g, \alpha).$$

Let $\varepsilon > 0$, then since $f$ and $g$ are Riemann-Stieltjes integrable, there exist partitions $P_1, P_2$ such that

$$U(P_1; f, \alpha) - L(P_1; f, \alpha) < \varepsilon/2, \quad U(P_2; g, \alpha) - L(P_2; g, \alpha) < \varepsilon/2.$$ 

If we let $P$ be a common refinement of $P_1$ and $P_2$, then by combining inequalities (1) and (2), we see that
\[ U(P; f+g, \alpha) - L(P; f+g, \alpha) \leq U(P; f, \alpha) - L(P; f, \alpha) + U(P; g, \alpha) - L(P; g, \alpha) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \]

**Property 3:** Set \( \gamma = \alpha + \beta \) and use condition (*) together with the fact that \( \Delta \gamma_i = \Delta \alpha_i + \Delta \beta_i \).

**Property 4:** This follows directly from the definition of the upper and lower integrals using, for example, the inequality \( M_i(g) \leq M_i(f) \).

**Property 5:** This is proved by applying property 4.) to the inequality

\[ -|f| \leq f \leq |f|, \]

from which it follows that

\[ -\int_a^b |f| \, d\alpha \leq \int_a^b f \, d\alpha \leq \int_a^b |f| \, d\alpha. \]

**Properties 6-7:** Use the properties above together with our earlier example.

\[ \square \]

**Theorem.** If \( f \) is Riemann-Stieltjes integrable with respect to \( \alpha \) on \([a,b] \), then it is Riemann integrable on each subinterval \([c,d] \subseteq [a,b] \). Moreover, if \( c \in [a,b] \), then

\[ \int_a^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha. \]

**Proof.** We set \( \alpha_1(x) = \alpha(x) \) on the interval \([a,c] \) and equal to the constant value \( \alpha(c) \) on \([c,b] \). Similarly, we set \( \alpha_2(x) = \alpha(x) - \alpha(c) \) if \( c \leq x \leq b \) and define it to vanish on \([a,c] \). Then the \( \alpha_j \) are non-decreasing and \( \alpha = \alpha_1 + \alpha_2 \). Note that \( \int_a^b f \, d\alpha_1 = \int_a^c f \, d\alpha \) and \( \int_a^b f \, d\alpha_2 = \int_c^b f \, d\alpha \). \( \square \)

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