ANALYSIS II

Riemann-Stieltjes Integration: Properties

Theorem. Suppose that α_1, α_2 are non-decreasing, and that f, g are Riemann-Stieltjes integrable with respect to both α_1 and α_2. If c is a nonnegative real number, then

1. $\int_a^b c\, f\ d\alpha = c \int_a^b f\, d\alpha = \int_a^b f\, d(c\,\alpha)$
2. $\int_a^b f+g\ d\alpha = \int_a^b f\, d\alpha + \int_a^b g\, d\alpha$
3. $\int_a^b f\ d(\alpha+\beta) = \int_a^b f\, d\alpha + \int_a^b f\, d\beta$
4. $g \le f$ implies $\int_a^b g\, d\alpha \le \int_a^b f\, d\alpha$.
5. $|\int_a^b f\, d\alpha| \le \int_a^b |f|\, d\alpha \le \|f\|_\infty (\alpha(b) - \alpha(a))$
6. $\int_a^b f\ d\alpha = f(s)$ if f is continuous at s between a and b, and $\alpha(x) = I(x-s)$ where $I(x) = 1$ for positive x and vanishes otherwise.
7. $\int_a^b f\ d\alpha = \sum_{n=1}^N c_n f(s_n)$ if $\alpha(x) = \sum_{n=1}^N c_n I(x-s_n)$, f is continuous at each of the s_n's (all of which are assumed to lie in the interval (a,b)), and the c_n's are nonnegative.

Proofs:

Property 1: We observe that $c \ge 0$ implies $c\,\alpha$ is nondecreasing, $M_i(c\,f) = c\,M_i(f)$ and $m_i(c\,f) = c\,m_i(f)$. Hence $U(P;cf,\alpha) = cU(P;f,\alpha) = U(P;f,c\alpha)$. A similar statement holds for lower sums.

Property 2: We notice that $M_i(f+g) \le M_i(f) + M_i(g)$ and $m_i(f) + m_i(g) \le m_i(f+g)$. Hence,

$$L(P;f,\alpha) + L(P;g,\alpha) \le L(P;f+g,\alpha) \le U(P;f+g,\alpha) \le U(P;f,\alpha) + U(P;g,\alpha).$$

Let $\varepsilon > 0$, then since f and g are Riemann-Stieltjes integrable, there exist partitions P_1,P_2 such that

$$U(P_1;f,\alpha)-L(P_1;f,\alpha) < \varepsilon/2, \quad U(P_2;g,\alpha)-L(P_2;g,\alpha) < \varepsilon/2.$$

If we let P be a common refinement of P_1 and P_2, then by combining inequalities (1) and (2), we see that

$$
\begin{aligned}
U(P; f+g,\alpha) - L(P; f+g,\alpha) &\leq U(P;f,\alpha)-L(P;f,\alpha) + U(P;g,\alpha)-L(P;g,\alpha) \\
&\leq U(P_1;f,\alpha)-L(P_1;f,\alpha) + U(P_2;g,\alpha)-L(P_2;g,\alpha) \\
&\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}
$$

Property 3: Set $\gamma = \alpha + \beta$ and use condition (*) together with the fact that $\Delta\gamma_i = \Delta\alpha_i + \Delta\beta_i$.

Property 4: This follows directly from the definition of the upper and lower integrals using, for example, the inequality $M_i(g) \leq M_i(f)$.

Property 5: This is proved by applying property 4.) to the inequality

$$-|f| \leq f \leq |f|,$$

from which it follows that

$-\int_a^b |f|\, d\alpha \leq \int_a^b f\, d\alpha \leq \int_a^b |f|\, d\alpha$.

Properties 6-7: Use the properties above together with our earlier example.

□

Theorem. If f is Riemann-Stieltjes integrable with respect to α on [a,b], then it is Riemann integrable on each subinterval $[c,d] \subseteq [a,b]$. Moreover, if $c \in [a,b]$, then

$$\int_a^b f\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha.$$

Proof. We set $\alpha_1(x) = \alpha(x)$ on the interval [a,c] and equal to the constant value $\alpha(c)$ on [c,b]. Similarly, we set $\alpha_2(x) = \alpha(x)-\alpha(c)$ if $c \leq x \leq b$ and define it to vanish on [a,c]. Then the α_j are non-decreasing and $\alpha = \alpha_1 + \alpha_2$. Note that $\int_a^b f\, d\alpha_1 = \int_a^c f\, d\alpha$ and $\int_a^b f\, d\alpha_2 = \int_c^b f\, d\alpha$. □

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