**Defn.** A collection of \( n+1 \) distinct points of the interval \([a,b]\)

\[ P: = \{x_0 := a < x_1 < \ldots < x_{i-1} < x_i < \ldots < b = : x_n\} \]

is called a **partition** of the interval. In this case, we define the **norm** of the partition by

\[ \|P\| := \max_{1 \leq i \leq n} \Delta x_i. \]

where \( \Delta x_i := x_i - x_{i-1} \) is the **length** of the \( i \)-th subinterval \([x_{i-1}, x_i]\).

For a non-decreasing function \( \alpha \) on \([a,b]\), define

\[ \Delta \alpha_i := \alpha(x_i) - \alpha(x_{i-1}). \]

**Defn.** Suppose that \( f \) is a bounded function on \([a,b]\) and \( \alpha \) is nondecreasing. For a given partition \( P \), we define the **Riemann-Stieltjes upper sum of a function \( f \) with respect to \( \alpha \)** by

\[ U(P;f,\alpha) := \sum_{i=1}^{n} M_i \Delta \alpha_i \]

where \( M_i \) denotes the supremum of \( f \) over each of the subintervals \([x_{i-1}, x_i]\). Similarly, we define the **Riemann-Stieltjes lower sum** by

\[ L(P;f,\alpha) := \sum_{i=1}^{n} m_i \Delta \alpha_i \]

where here \( m_i \) denotes the infimum of \( f \) over each of the subintervals \([x_{i-1}, x_i]\). Since \( m_i \leq M_i \) and \( \Delta \alpha_i \) is nonnegative, we observe that
\[ L(P; f, \alpha) \leq U(P; f, \alpha). \]

for any partition \( P \).

**Defn.** Suppose \( P_1, P_2 \) are both partitions of \([a,b]\), then \( P_2 \) is called a *refinement* of \( P_1 \) (denoted by \( P_1 \leq P_2 \)) if as sets \( P_1 \subseteq P_2 \).

**Note.** If \( P_1 \leq P_2 \), it follows that \( ||P_2|| \leq ||P_1|| \) since each of the subintervals formed by \( P_2 \) is contained in a subinterval which arises from \( P_1 \).

**Lemma.** If \( P_1 \leq P_2 \), then

\[ L(P_1; f, \alpha) \leq L(P_2; f, \alpha). \]

and

\[ U(P_2; f, \alpha) \leq U(P_1; f, \alpha). \]

**Pf.** Suppose first that \( P_1 \) is a partition of \([a,b]\) and that \( P_2 \) is the partition obtained from \( P_1 \) by adding an additional point \( z \). The general case follows by induction, adding one point at a time. In particular, we let

\[ P_1 := \{ x_0 := a < x_1 < ... < x_{i-1} < x_i < ... < b = : x_n \} \]

and

\[ P_2 := \{ x_0 := a < x_1 < ... < x_{i-1} < z < x_i < ... < b = : x_n \} \]

for some fixed \( i \). We focus on the upper sum for these two partitions, noting that the inequality for the lower sums follows similarly. Observe that

\[ U(P_1; f, \alpha) := \sum_{j=1}^{n} M_j \Delta \alpha_j \]

and
\[ U(P_2;f,\alpha) = \sum_{j=1}^{i-1} M_j \Delta \alpha_j + M (\alpha(z) - \alpha(x_{i-1})) + \sum_{j=i+1}^{n} M_j \Delta \alpha_j \]

where \( M := \sup \{ f(x) | x_{i-1} \leq x \leq z \} \) and \( M^\sim := \sup \{ f(x) | z \leq x \leq x_i \} \). It then follows that \( U(P_2;f,\alpha) \leq U(P_1;f,\alpha) \) since

\[ M, M^\sim \leq M_i. \]

**Defn.** If \( P_1 \) and \( P_2 \) are arbitrary partitions of \([a,b]\), then the **common refinement** of \( P_1 \) and \( P_2 \) is the formal union of the two.

**Corollary.** Suppose \( P_1 \) and \( P_2 \) are arbitrary partitions of \([a,b]\), then

\[ L(P_1;f,\alpha) \leq U(P_2;f,\alpha). \]

**Pf.** Let \( P \) be the common refinement of \( P_1 \) and \( P_2 \), then

\[ L(P_1;f,\alpha) \leq L(P;f,\alpha) \leq U(P;f,\alpha) \leq U(P_2;f,\alpha). \]

**Defn.** The **lower Riemann-Stieltjes integral** of \( f \) with respect to \( \alpha \) over \([a,b]\) is defined to be

\[ (L) - \int_a^b f(x) \, d\alpha := \sup_{\text{all partitions } P \text{ of } [a,b]} L(P;f,\alpha). \]

Similarly, the **upper Riemann-Stieltjes integral** of \( f \) with respect to \( \alpha \) over \([a,b]\) is defined to be

\[ (U) - \int_a^b f(x) \, d\alpha(x) := \inf_{\text{all partitions } P \text{ of } [a,b]} U(P;f,\alpha). \]

By the definitions of least upper bound and greatest lower bound, it is evident that for any function \( f \) there holds
\[
(L) - \int_a^b f(x) \, d\alpha(x) \leq (U) - \int_a^b f(x) \, d\alpha(x).
\]

**Defn.** A function \( f \) is *Riemann-Stieltjes integrable* over \([a,b]\) if the upper and lower Riemann-Stieltjes integrals coincide. We denote this common value by \( \int_a^b f(x) \, d\alpha(x) \).

**Examples:**

1. Obviously, if \( \alpha(x) := x \), then the Riemann-Stieltjes integral reduces to the Riemann integral of \( f \).
2. \( \int_a^b f(x) \, d\alpha(x) = f(x_0) \), if \( f \) is continuous at \( x_0 \) and \( \alpha \) is defined to be the step function which is one for \( x \) larger than \( x_0 \) and zero otherwise.

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