ANALYSIS II
Riemann-Stieltjes Integration: Conditions for Existence

In the previous section, we saw that it was possible for \( \alpha \) to be discontinuous but for the Reiemann-Stieltjes integral of \( f \) to still exist. The following example shows that the integral may not exist however, if both \( f \) and \( \alpha \) are discontinuous at a point.

**Example.** Let \( f = \alpha \) where \( f(x) \) is one for nonnegative \( x \) and zero otherwise. In this case, if \( P \) is any partition, \( U(P;f,\alpha) = 1 \), while \( L(P;f,\alpha) = 0 \). This shows that the Riemann-Stieltjes integral for this pair does not exist.

**Theorem.** A necessary and sufficient condition for \( f \) to be Riemann-Stieltjes integrable with respect to \( \alpha \) is for each given \( \varepsilon > 0 \), that one can obtain a partition \( P \) of \([a,b]\) such that

\[
U(P;f,\alpha) - L(P;f,\alpha) < \varepsilon.
\]

**Pf.** First we show that (*) is a sufficient condition. This follows immediately, since for each \( \varepsilon > 0 \) that there is a partition \( P \) such that (*) holds,

\[
(U) \int_a^b f(x) \, d\alpha(x) - (L) \int_a^b f(x) \, d\alpha(x) \leq U(P;f,\alpha) - L(P;f,\alpha) < \varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, then the upper and lower Riemann-Stieltjes integrals of \( f \) must coincide.

To prove that (*) is a necessary condition for \( f \) to be Riemann integrable, we let \( \varepsilon > 0 \). By the definition of the upper Riemann-Stieltjes integral as a infimum of upper sums, we can find a partition \( P_1 \) of \([a,b]\) such that

\[
\int_a^b f(x) \, d\alpha(x) \leq U(P_1;f,\alpha) < \int_a^b f(x) \, d\alpha(x) + \varepsilon/2.
\]

Similarly, we have

\[
\int_a^b f(x) \, d\alpha(x) - \varepsilon/2 < L(P_2;f,\alpha) \leq \int_a^b f(x) \, d\alpha(x).
\]
Let $P$ be a common refinement of $P_1$ and $P_2$, then subtracting the two previous inequalities implies,

$$U(P;f,\alpha) - L(P;f,\alpha) \leq U(P_1;f,\alpha) - L(P_2;f,\alpha) < \varepsilon. \quad \square$$

**Theorem.** If $f$ is continuous on $[a,b]$, then $f$ is Riemann-Stieltjes integrable with respect to $\alpha$ on $[a,b]$.

**Pf.** We use the condition (*) to establish the proof. If $\varepsilon > 0$, we set $\varepsilon_0 := \varepsilon/(1+\alpha(b)-\alpha(a))$. Since $f$ is continuous on $[a,b]$, $f$ is uniformly continuous. Hence there is a $\delta > 0$ such that $|f(y)-f(x)| < \varepsilon_0$ if $|y-x| < \delta$. Suppose that $\|P\| < \delta$, then it follows that $|M_i - m_i| < \varepsilon_0$ $(1 \leq i \leq n)$. Hence

$$U(P;f,\alpha) - L(P;f,\alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha_i < \varepsilon_0 (\alpha(b)-\alpha(a)) < \varepsilon. \quad \square$$

**Theorem.** If $f$ is monotone and $\alpha$ is continuous on $[a,b]$, then $f$ is Riemann-Stieltjes integrable with respect to $\alpha$ on $[a,b]$.

**Pf.** We prove the case for $f$ monotone increasing and note that the case for monotone decreasing is similar. We again use the condition (*) to prove the theorem. If $\varepsilon > 0$, we set $\varepsilon_0 := \varepsilon/(1+f(b)-f(a))$, Since $\alpha$ is continuous and $[a,b]$ is compact, $\alpha$ is uniformly continuous. So for $\varepsilon_0$ we can determine a $\delta > 0$, so that if $P$ is a partition with $\|P\| < \delta$, then $\Delta \alpha_i < \varepsilon_0$ (all $i$). The function $f$ is monotone increasing on $[a,b]$, so $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Hence

$$U(P;f,\alpha) - L(P;f,\alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha_i$$

$$= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \Delta \alpha_i$$

$$< \varepsilon_0 \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))$$

$$\leq \varepsilon_0 (f(b)-f(a)) < \varepsilon. \quad \square$$

**Defn.** A Riemann-Stieltjes sum for $f$ with respect to $\alpha$ for a partition $P$ of an interval $[a,b]$ is defined by
\[ R(P;\xi) = \sum_{j=1}^{n} f(\xi_j) \Delta \alpha_j \]

where the \( \xi_j \), satisfying \( x_{j-1} \leq \xi_j \leq x_j \) \((1 \leq j \leq n)\), are arbitrary.

**Corollary.** Suppose that \( f \) is Riemann-Stieltjes integrable on \([a,b]\), then there is a unique number \( \gamma (= \int_{a}^{b} f \, d\alpha) \) such that for every \( \varepsilon > 0 \) there exists a partition \( P \) of \([a,b]\) such that if \( P \leq P_1, P_2 \), then

\[ 
\begin{align*}
\text{i.)} \quad & 0 \leq U(P_1;f,\alpha) - \gamma < \varepsilon \\
\text{ii.)} \quad & 0 \leq \gamma - L(P_2;f,\alpha) < \varepsilon \\
\text{iii.)} \quad & |\gamma - R(P_1,\xi)| < \varepsilon \\
\end{align*}
\]

where \( R(P_1,\xi) \) is any Riemann-Stieltjes sum of \( f \) with respect to \( \alpha \) for the partition \( P_1 \). In this case, we can interpret the integral as

\[ \int_{a}^{b} f \, d\alpha = \lim_{\|P\| \to 0} R(P,\xi), \]

although a careful proof is somewhat involved.

**Pf.** Since \( L(P_2;f,\alpha) \leq \gamma \leq U(P_1;f,\alpha) \) for all partitions, we see that parts i.) and ii.) follow from the definition of the integral. To see part iii.), we observe that \( m_j \leq f(\xi_j) \leq M_j \) and hence that

\[ L(P_1;f,\alpha) \leq R(P_1,\xi) \leq U(P_1;f,\alpha). \]

But we also know that both

\[ L(P_1;f,\alpha) \leq \gamma \leq U(P_1;f,\alpha) \]

and condition (*) hold, from which part iii.) follows. \( \square \)