ANALYSIS II

Metric Spaces: Limits and Continuity

**Defn** Suppose \((X,d)\) is a metric space and \(A\) is a subset of \(X\).

1. A point \(x\) is called an *interior point* of \(A\) if there is a neighborhood of \(x\) contained in \(A\).
2. A set \(N\) is called a *neighborhood (nbhd) of \(x\)* if \(x\) is an interior point of \(N\).
3. A point \(x\) is called a *boundary point* of \(A\) if it is a limit point of both \(A\) and its complement.
4. A point \(x\) is called a *limit point* of the set \(A\) if each neighborhood of \(x\) contains points of \(A\) distinct from \(x\).
   (This is equivalent to saying that each neighborhood of \(x\) has an infinite number of members of \(A\). Recall that a neighborhood for a point \(x\), is a set containing an open \(\epsilon\)-nbhd of \(x\).)
5. A point \(x\) is called an *isolated point* of \(A\) if \(x\) belongs to \(A\) but is not a limit point of \(A\).

**Proposition** A set \(O\) in a metric space is open if and only if each of its points are interior points.

**Proposition** A set \(C\) in a metric space is closed if and only if it contains all its limit points.

**Defn** Suppose \((X,d)\) is a metric space and \(A\) is a subset of \(X\). The *closure of \(A\)* is the smallest closed subset of \(X\) which contains \(A\). The *derived set \(A'\)* of \(A\) is the set of all limit points of \(A\).

**Proposition** The closure of \(A\) may be determined by either

- the intersection of all closed sets which contain \(A\),

or

- the union of \(A\) with its derived set.

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**Sequential Convergence**

**Defn** A sequence \(\{x_n\}\) in a metric space \((X,d)\) is said to *converge*, to a point \(x_0\) say, if for each neighborhood of \(x_0\) there exists a natural number \(N\) so that \(x_n\) belongs to the neighborhood if \(n\) is greater or equal to \(N\); that is, eventually the sequence is contained in the neighborhood. In this case, we say that \(x_0\) is the *limit of the sequence* and write
\[ \lim_{n \to \infty} x_n := x_0. \]

**Proposition** In a metric space, sequential limits are unique.

**Proposition** That a sequence \( \{x_n\} \) converges in a metric space \((X,d)\) to a point \(x_0\) is equivalent to the condition that for each \(\varepsilon > 0\) there is a natural number \(N\) such that \(N \leq n\) implies \(d(x_n, x_0) < \varepsilon\).

**Examples**

1. In either the reals or complexes if \(|r| < 1\), then \(r^n \to 0\).
2. Consider the space of continuous functions on \([0,1/2]\), \(C[0,1/2]\). Let \(f_n(x) = x^n\), then \(f_n \to 0\).
3. The sequence \(f_n(x) = x^n\) belongs to \(C[0,1]\) but does not converge.

**Defn** A function \(f\) defined on \(X\setminus\{x_0\}\), with values in a metric space \(\{Y,d_2\}\) is said to have a limit \(L\) at \(x_0\) if \(x_0\) is a limit point of \(X\) and for each neighborhood \(O_2\) of \(L\), there is a neighborhood \(O_1\) of \(x_0\) such that \(f\) maps each element of the deleted neighborhood \(O_1\setminus\{x_0\}\) into \(O_2\). This is denoted

\[ \lim_{x \to x_0} f(x) := L. \]

**Homework** This is equivalent to the condition: for each \(\varepsilon > 0\) there is a \(\delta > 0\) such that if \(0 < d_1(x,x_0) < \delta\), then \(d_2(f(x),L) < \varepsilon\).

**Proposition** A necessary and sufficient condition for a function \(f\) to have a limit \(L\) at \(x_0\) is that for each sequence \(\{x_n\}\) which converges to \(x_0\) (no point of which is equal to \(x_0\)), then \(\{f(x_n)\}\) converges to \(L\). Consequently, if a function has a limit at a point \(x_0\), then it is unique.

**Defn** A function \(f\) is called continuous at a point \(x_0\) if either

1. \(x_0\) is an isolated point of \(X\) or
2. \(x_0\) is a limit point of \(X\) and the limit of \(f\) as \(x\) approaches \(x_0\) is \(f(x_0)\).

**Homework** A necessary and sufficient condition for a function \(f\) to be continuous at \(x_0\) is that for each \(\varepsilon > 0\) there is a \(\delta > 0\) such that if \(d_1(x,x_0) < \delta\), then \(d_2(f(x),f(x_0)) < \varepsilon\).
**Continuity**

**Defn** Suppose \( f : X \to Y \) where \((X,d_1)\) and \((Y,d_2)\) are metric spaces. \( f \) is called **continuous** if the inverse image of each open set in \( Y \) is open in \( X \).

**Proposition** A function \( f : X \to Y \) is continuous if and only if the inverse image of each closed set in \( Y \) is closed in \( X \).

**Theorem** A function \( f : X \to Y \) is continuous if and only if \( f \) is continuous at each point of \( X \).

**Theorem** Suppose that \( f : X \to Y \) and \( g : Y \to Z \) are continuous functions, then \( g \circ f \) is a continuous function from \( X \) to \( Z \).

**Theorem** Suppose that \((X,d_X)\) and \((Y,d_Y)\) are both metric spaces, then \( X \times Y \) is a metric space if the metric \( d \) is defined for \( z_i = (x_i,y_i), \ i=1,2, \) by

\[
d(z_1,z_2) := d_X(x_1,x_2) + d_Y(y_1,y_2).
\]

**Examples:**

1. For a metric space \((X,d)\), the metric \( d \) is a continuous function from \( X^2 \) to \( \mathbb{R} \).
2. Suppose that \((X,\|\cdot\|)\) is a normed linear space, then both the vector space operations are jointly continuous:
   1. if \( a_n \to a \) in \( \mathbb{R} \) and \( x_n \to x \) in \( X \), then \( \|a_n x_n\|_X \to \|a x\|_X \) in \( \mathbb{R} \).
   2. if \( x_n \to x \) and \( y_n \to y \) in \( X \), then \( x_n + y_n \to x + y \) in \( X \).

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