ANALYSIS II
Introduction
Metric and Normed Linear Spaces

Defn A ***metric space*** is a pair (X,d) where **X** is a set and **d** : $X^2 \to [0,\infty)$ with the properties that, for each x,y,z in X:

1. d(x,y)=0 if and only if x=y,
2. d(x,y) = d(y,x),
3. $d(x,y) \leq d(x,z) + d(z,y)$.

d is called the ***distance function*** and d(x,y) denotes the ***distance between x and y***.

Note: A given set X may be measured by various distances in order to study the set in different ways.

Examples

- X is any set and d(x,y) := 1 if and only if x is not y.
- The real numbers with absolute value: i.e., X = **R** and d(x,y) := |x-y|.
- The complex numbers with modulus: i.e. X = **C** and $d(z_1, z_2) := |z_1 - z_2|$.
- (X, ||·||) a normed linear space (see below) and d(x,y) := ||x-y||. (Verify!)
- $X = \mathbf{R}^k$ and for $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ define
  1. the standard ***Euclidean*** distance as $d_2(\mathbf{x},\mathbf{y}) := (\sum_i |x_i - y_i|^2)^{1/2}$
  2. $d_p(\mathbf{x},\mathbf{y}) := (\sum_i |x_i - y_i|^p)^{1/p}, \ 1 \leq p$
  3. $d_\infty(\mathbf{x},\mathbf{y}) := \max_i |x_i - y_i|$

  We will concentrate our studies on the cases $p=1,2,\infty$. We prove that each of the above are metric spaces by showing that they are normed linear spaces, where the obvious candidates are used for norms. The following metrics do not arise as norms [otherwise we must have $d(a\,\mathbf{x}, a\,\mathbf{y}) = |a|\, d(\mathbf{x},\mathbf{y})$].

  1. $d_p(\mathbf{x},\mathbf{y}) := (\sum_i |x_i - y_i|^p), \ 0 < p \leq 1$
  2. X = **R** and d(x,y) := |x-y|/(1 + |x-y|)
     (Homework)

Defn A ***normed linear space*** is a vector space X and a non-negative valued mapping ||·|| on X, called the ***norm***, which satisfies the properties

1. $||x||=0$ if and only if $x=0$.
2. $||a\, x|| = |a|\, ||x||$, for all scalars a.
3. $||x+y|| \le ||x|| + ||y||$

Here $||\mathbf{x}||$ is thought of as the ***length of x*** or the distance from **x** to **0**. Notice that for a given vector **x**, if **y** is defined as $(1/||x||)\,\mathbf{x}$, then **y** has unit length and is called the ***normalized*** vector for **x**.

Examples

- $X = \mathbf{R}$ and $||x|| := |x|$.
- $X = \mathbf{C}$. For z in **C**, the modulus of z, $|z| := (|\mathrm{Re}\, z|^2 + |\mathrm{Im}\, z|^2)^{1/2}$ is a norm for the complex numbers.
- $X = \mathbf{R}^k$ and for $\mathbf{x} = (x_1, x_2, ..., x_k)$ define

1. the standard ***Euclidean*** distance as $||x||_2 := (\sum_i |x_i|^2)^{1/2}$.

   Pf: Although this is a special case of the p-norms, it is instructive to demonstrate this separately: First we establish

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   Lemma (Cauchy-Schwarz Inequality)
   $(\sum_i |x_i\, y_i|) \le ||\mathbf{x}||_2\ ||\mathbf{y}||_2$.

   Pf: We may assume, without loss of generality, that neither x nor y are the zero vector. First assume that x and y are unit vectors, i.e. $||x||=||y||=1$. Observe by expanding out the square that the inequality
   $|ab| \le (a^2+b^2)/2$

   holds. (This is essentially the famous arithemetic-geometric inequality (using $x=a^2$ and $y=b^2$).) It then follows that
   $(\sum_i |x_i\, y_i|) \le 1/2 \sum_i (|x_i|^2 + |y_i|^2) = 1$
   proving Cauchy's inequality in the special case of unit vectors. For general nonzero vectors, apply this inequality to the normalized vectors for x and y. □

   Continuing the proof that $||.||_2$ is a norm, we observe

   $$(\sum_i |x_i + y_i|^2) \le ||x||^2 + ||y||^2 + 2 (\sum_i |x_i\, y_i|) \le ||x||^2 + ||y||^2 + 2\, ||\mathbf{x}||_2\ ||\mathbf{y}||_2 = (||x||_2 + ||y||_2)^2$$

   and complete the proof by taking square roots. □
2. $||x||_p := (\sum_i |x_i|^p)^{1/p}$, $1 \le p$

   The subadditivity of this norm is known as Minkowski's inequality and relys on

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   Lemma (Holder's Inequality) Suppose that $1/p+1/q =1$ where $1 \le p$, then
   $(\sum_i |x_i\, y_i|) \le ||\mathbf{x}||_p\ ||\mathbf{y}||_q$.

   Pf: We first show that for nonnegative a and b, that
   (*) $a^c b^d \le c\, a + d\, b$
   is true where we have set $c = 1/p$ and $d = 1-c = 1/q$. Notice that the statement is symmetric in p and q (that is they can

be interchanged in the statemnet of the Lemma). If a and b are nonnegative real numbers, then we may assume without loss of generality that neither vanishes. We may also assume by the symmetry of p and q that a is larger than b. Observe by the first derivative test that the function $f(u)=1+u-u^c$. is a strictly monotone increasing function for u greater or equal 1. This shows in particular that $f(a/b)>0$ if $a/b>1$ since $f(1)=0$. The inequality (*) follows by multiplying through by b. We apply (*) to $a = x^p$ and $b = y^q$ to obtain for nonnegative numbers x and y

$$(**) \quad x\,y \le x^p/p + y^q/q$$

Hence it follows that if **x** and **y** are normalized vectors ($\|x\|_p = \|y\|_q = 1$), then

$$\left(\sum_i |x_i\, y_i| \right) \le 1/p \left(\sum_i |x_i|^p \right) + 1/q \left(\sum_i |y_i|^q \right) = 1.$$

Holder's inequality follows by normalizing general nonzero vectors as was done in the Cauchy-Schwarz inequality. □

Pf: To prove the subadditivity of the p-norm, we let $z_i := |x_i+y_i|^{p-1}$. Then

$$\|x+y\|_p^p \le \left(\sum_i |x_i|\, z_i \right) + \left(\sum_i |y_i|\, z_i \right) \le (\|x\|_p + \|y\|_p)\, \|z\|_q$$

where in the second inequality from the left, Holder's inequality is applied to both ($|x_i|$, z_i) and ($|y_i|$, z_i). The inequality follows by noticing that

$$\|z\|_q = \|x+y\|_p^{p/q}$$

and that $p/q = p-1$.

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1. $\|x\|_\infty := \max_i |x_i|$

   Pf: The first two properties of norm follow directly from the properties of absolute value. To establish the subadditve property (3), we observe that for each i between 1 and k,

   $$|(x+y)_i| \le |x_i| + |y_i| \le \|x\| + \|y\|$$

   and then take the maximum over i.

Defn **C[a,b]** is the set of continuous functions on [a,b]. If f belongs to C[a,b], then $\|f\|_\infty := \max_x |f(x)|$ is defined as the norm of f. Sometimes, this is referred to as the ***sup norm***.

(Note that the max is always attained in the norm by the extreme value theorem.}

Proposition C[a,b] is a normed linear space.

Pf: By the properties of continuous functions, C[a,b] is a vector space.Since $|f(x)+g(x)| \le \|f\| + \|g\|$ for all x in [a,b], taking the maximum over all such x, the subadditivity of the norm is established.

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