ANALYSIS II

Metric Spaces: Completeness

Defn Suppose (X,d) is a metric space. A sequence $\{x_n\}$ is said to a be ***Cauchy sequence*** in (X,d) if, for each $\epsilon > 0$, there is a natural number N such that $d(x_n, x_m) < \epsilon$ if $N \leq n,m$.

Proposition Convergent sequences are Cauchy.

Lemma Cauchy sequences are bounded, but not necessarily convergent.

Pf: Consider X as the interval (0,1] with the absolute value as norm. The sequence $\{x_n\}$ with $x_n = 1/n$ is Cauchy but not convergent in X. To show that each Cauchy sequence is bounded, apply the definition with $\epsilon := 1$ to obtain an N such that $d(x_n, x_m) < 1$ if $N \leq n,m$. Let $R := \max\{1, d(x_N, x_1), d(x_N, x_2), \ldots, d(x_N, x_{N-1})\}$, then $\{x_n\}$ is contained in $B_{2R}(x_N)$.

Defn If (X,d) is a metric space for which each Cauchy sequence converges, then (X,d) is said to be a ***complete metric space***.

Lemma If a subsequence of a Cauchy sequence converges, then the sequence is itself convergent to the same limit.

Proposition If C is a closed subset of a complete metric space (X,d), then C is a complete metric space with the restricted metric.

Examples **R, C, $\mathbf{R}^k$, $\mathbf{C}^k$** are all complete metric spaces.
Pf: Use the fact that convergence of a sequence in each of the spaces **C, $\mathbf{R}^k$, $\mathbf{C}^k$** is equivalent to convergence in each coordinate.of the sequence. This follows since the sup norm is equivalent to the Euclidean norm.

Theorem Let C[a,b] denote the normed linear space of continuous functions on the interval [a,b] equipped (as before) with the sup-norm, then C[a,b] is complete.

Pf: Let $\{f_n\}$ be a Cauchy sequence in C[a,b].
Step 1 Establish a pointwise limit for $\{f_n\}$and call this function f.
Step 2: Prove that $\| f_n - f \|_\infty \to 0$.
Step 3: Next show that the function f is continuous on [a,b]. (Hint: Use an $\epsilon/3$ argument.)

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