Defn. A function $f$ is said to be differentiable at $x_0$ if
\[
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}
\]
exists. In this case the limit is called the derivative of $f$ at $x_0$ and is denoted $f'(x_0)$.

Note. 1. This definition is equivalent to the requirement that the following limit exist:
\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).
\]
2. This, in turn, is equivalent to the following statement about how fast $f(x)$ converges to $f(x_0)$ as $x \to x_0$:
there exists a function $\eta$ such that \( \lim_{x \to x_0} \eta(x) = 0 \) and
\[
(*) \quad f(x) - f(x_0) = (x - x_0) (f'(x_0) + \eta(x)).
\]

Examples: 1. If $f(x) := x^2$, then $f'(x) = 2x$.
2. If $g(x) := |x|$, then $g'(0)$ does not exist.
3. If $h(x) := x|x|$, then $h'(x)$ exists and equals $2|x|$.

Theorem. If $f$ is differentiable at $x_0$, then $f$ is continuous at $x_0$.
Proof. Use (*) and let $x \to x_0$. \( \square \)

Theorem. (Basic rules of differentiation: sums, products, quotients) Suppose that $f$ and $g$ are differentiable at $x_0$, then
1. $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.
2. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.
3. $(f/g)'(x_0) = (g(x_0)f'(x_0) - f(x_0)g'(x_0)) / g(x_0)^2$, if $g(x_0) \neq 0$.

Theorem. (Chain rule) If $f$ is differentiable at $x_0$ and $g$ is differentiable at $y_0 := f(x_0)$, then $h := g \circ f$ is differentiable at $x_0$ and
\[
h'(x_0) = g'(f(x_0)) f'(x_0)
\]
**Proof.** Use (*) for $f$ at $x_0$ and for $g$ at $y_0 := f(x_0)$:

$$
\frac{h(x) - h(x_0)}{x - x_0} = \frac{g(y) - g(y_0)}{x - x_0} = \frac{y - y_0}{x - x_0}(g'(y_0) + \eta_2(y))
$$

$$
= \frac{f(x) - f(x_0)}{x - x_0}(g'(y_0) + \eta_2(y))
$$

$$
= (f'(x_0) + \eta_1(x))(g'(y_0) + \eta_2(y))
$$

where $y := f(x)$. The proof is completed by using this equation, letting $x_n \to x_0$, and noticing that $y_n \to y_0$ where $y_n := f(x_n)$. \qed

**Theorem.** (Rolle’s Theorem) Suppose that $\phi$ is differentiable on $(a,b)$, is continuous on $[a,b]$, and vanishes at the endpoints, then there exists $x_0$ strictly between $a$ and $b$ such that $\phi'(x_0) = 0$.

**Proof.** If $\phi$ is constant, then any point can be selected for $x_0$. Otherwise, we may assume WLOG that $\phi$ has positive values. By the Extreme Value Theorem, let $x_0$ be such that $\phi(x) \leq \phi(x_0)$ for all $a \leq x \leq b$. First, let $x_n \downarrow x_0$, then since $x_0$ gives a max, we have

$$
0 \geq \frac{\phi(x_n) - \phi(x_0)}{x_n - x_0} \to \phi'(x_0)
$$

and so, by the Squeeze Theorem, $\phi'(x_0) \leq 0$. Similarly, $\phi'(x_0) \geq 0$. \qed

**Note.** Within the proof we actually established the critical point procedure of calculus: local max and min can only occur at critical points.

**Corollary.** (Mean Value Theorem) Suppose that $f$ is differentiable on $(a,b)$ and is continuous on $[a,b]$, then there exists $x_0$ strictly between $a$ and $b$ such that

$$
f'(x_0) = \frac{f(b) - f(a)}{b - a}.
$$

**Proof.** Let

$$
\phi(x) := f(x) - \left[ \frac{f(b)-f(a)}{b-a} (x - a) + f(a) \right]
$$

and apply Rolle’s theorem. \qed

**Defn.** $F$ is called an anti-derivative of $f$ if $F$ is differentiable and $F'(x) = f(x)$

**Corollary.** If both $F$ and $G$ are anti-derivatives of $f$, then they differ by a constant, i.e. there exists a constant $c$ such that $F(x) - G(x) = c$, for all $x \in dom(f)$. 