**Defn.** Suppose that \( K \subseteq \mathbb{R} \). A collection \( G \) of open subsets such that
\[
K \subseteq \bigcup_{O \in G} O.
\]
is called an *open cover* of \( K \). \( K \) has a *finite subcover* from \( G \) if there exist \( O_1, O_2, \ldots, O_n \) in \( G \) for which
\[
K \subseteq \bigcup_{j=1}^{n} O_j.
\]

**Defn.** \( K \) is called *compact*, if each open cover \( G \) of \( K \) has a finite subcover.

**Theorem.** The continuous image of a compact set is compact.

*Proof.* Suppose \( f : K \to \mathbb{R} \) is continuous and \( K \) is compact. Each open cover \( C \) of \( f[K] \) can be drawn back to an open cover \( \tilde{C} \) of \( K \), by considering the sets
\[
\tilde{O} := f^{-1}(O), \ O \in C.
\]

\( K \) compact implies that we may draw a finite subcover from \( \tilde{C} \). Each of these members is the inverse image (under \( f \)) from a member of \( C \). These form the desired subcover of \( f[K] \). \( \square \)

**Theorem.** (Heine-Borel) Suppose that \( a \leq b \), then the interval \([a,b]\) is compact.

*Proof.* Let \( C \) be an open cover for \([a,b]\) and consider the set
\[
A := \{x | [a,x] \text{ has an open cover from } C\}.
\]
Note that \( A \neq \emptyset \) since \( a \in A \). Let \( \gamma := \text{lub}(A) \). It is enough to show that \( \gamma > b \), since if \( x_1 \in A \) and \( a \leq x \leq x_1 \), then \( x \in A \). Suppose instead that \( \gamma \leq b \), then there must be some \( O_0 \in C \) such that \( \gamma \in O_0 \). But \( O_0 \) is open, so there exists \( \delta > 0 \) so that \( N_{\delta}(\gamma) \subseteq O_0 \). Since \( \gamma \) is the least upper bound for \( A \), then there is an \( x \in A \) such that \( \gamma - \delta < x \leq \gamma \). But \( x \in A \) implies there are members \( O_1, \ldots, O_n \) of \( C \) whose union covers \([a,x]\). The collection \( O_0, O_1, \ldots, O_n \) covers \([a, \gamma + \delta/2]\). Contradiction, since \( \gamma \) is the least upper bound for the set \( A \). \( \square \)

**Theorem.** Each closed subset \( C \) of a compact set \( K \) is compact.

*Proof.* Let \( \tilde{G} \) be an open cover for \( C \). Let \( O_0 \) be the complement of \( C \), then \( O_0 \) is open and \( G := \tilde{G} \cup \{O_0\} \) is an open cover for \( K \). There is a finite subcover of \( G \) which covers \( K \) and hence \( C \). This subcover (dropping \( O_0 \) if it appears) is the desired finite subcover for \( C \). \( \square \)
**Defn.** Suppose \( \{a_n\} \) is a sequence. A sequence \( \{b_k\} \) is called a subsequence of \( \{a_n\} \) if there exists a strictly increasing sequence of natural numbers \( n_1 < n_2 < \ldots < n_k < \ldots \) such that \( b_k = a_{n_k}, \ k = 1, 2, \ldots \)

**Theorem.** Suppose that \( K \subseteq \mathbb{R} \), then TFAE:

a.) \( K \) is compact,

b.) \( K \) is closed and bounded,

c.) each sequence in \( K \) has a subsequence which converges to a member of \( K \),

d.) (Bolzano-Weierstrass) each infinite subset of \( K \) has a limit point in \( K \).

**Proof.** (a) \( \Rightarrow \) (b) : To show that \( K \) is bounded, consider the open cover of \( K \) consisting of the collection of nested open intervals \( \mathcal{O}_n := (-n, n), \ n \in \mathbb{N} \). To show that \( K \) is closed, let \( x_0 \) be a limit point of \( K \). Assume to the contrary that \( x_0 \notin K \). Consider the open cover of \( K \) consisting of the collection of nested open sets \( \mathcal{O}_n := \{x \in \mathbb{R}| |x - x_0| > 1/n\}, \ n \in \mathbb{N} \). Any finite subcollection which would cover \( K \) would have union whose complement would be a neighborhood of \( x_0 \) not intersecting \( K \). This shows that \( x_0 \) could not be a limit point of \( K \).

(b) \( \Rightarrow \) (d) : We use the ‘divide and conquer’ method, better known as the ‘bisection’ method. Let \( A \) be an infinite subset of \( K \). Since \( K \) is bounded, there is an interval \([a, b]\) such that \( K \subseteq [a, b] \). Inductively define the closed subintervals as follows. Let \([a_0, b_0] := [a, b] \). Either the left or right half of \([a_0, b_0]\) contains an infinite number of members of \( K \). In the case that it is the right half, set \([a_1, b_1] := [(b_0 + a_0)/2, b_0] \). Set \([a_1, b_1]\) equal to the left half of \([a_0, b_0]\) otherwise. Inductively, let \([a_{n+1}, b_{n+1}]\) be the half of \([a_n, b_n]\) which contains an infinite number of members of \( A \). Notice that the length of this interval is \((b - a)/2^{n+1}\), that the \( a_n \)'s satisfy \( a_n \leq a_{n+1} \leq \ldots < b \) and so must converge to some real number \( a \leq x_0 \leq b \). Each neighborhood of \( x_0 \) will contain one of the intervals \([a_n, b_n]\) and hence will contain an infinite number of members of \( A \), i.e. \( x_0 \) is a limit point of \( A \). This also shows that \( x_0 \) is a limit point of the closed set \( K \) and must therefore belong to \( K \).

(d) \( \Rightarrow \) (c) : Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence in \( K \). If the sequence’s image is finite, then we may construct a constant subsequence which has the value which we may choose as any of the values of \( \{x_n\}_{n=1}^{\infty} \) which is repeated infinitely often. Otherwise, let \( A \) be the range of the sequence. Then \( A \) is an infinite subset of \( K \). By the Bolzano-Weierstrass property, \( A \) must have a limit point (\( x_0 \) say) which belongs to \( K \). For each \( k \in \mathbb{N} \), we may find an integer \( n_k \) larger than those previously picked (i.e., \( n_1, \ldots, n_{k-1} \)), so that \( |x_{n_k} - x_0| < 1/k \). This is the desired subsequence.
(c) ⇒ (b): If $K$ were not bounded, then there would exist a sequence $x_n \in K$ such that $|x_n| > n$. If this sequence had a subsequence which converged, then it would have to be bounded. But each subsequence of $\{x_n\}$ is clearly unbounded. To show that $K$ is closed, we let $x_0$ be a limit point of $K$ which is not in $K$. We can then find a sequence $\{x_n\}$ from $K$ which converges to $x_0$. By condition (c), this has to have a subsequence which converges to a member of $K$. Contradiction. Each subsequence of a convergent sequence converges to the same limit, in this case $x_0$, which does not belong to $K$. □

**Corollary.** Each continuous function $f$ on a compact set $K$ is bounded.
*Proof. The set $f(K)$ is compact and is therefore bounded. □*

**Corollary.** (Extreme Value Theorem) Each continuous function on a compact set attains its maximum (resp. minimum).
*Proof. The set $f(K)$ is compact and is therefore bounded and closed. Hence the least upper bound $\gamma$ for $f(K)$ must belong to $f(K)$. Therefore, there is an $x_0 \in K$ such that $\gamma = f(x_0)$ and so

$$f(x) \leq f(x_0), \text{ for all } x \in K.$$ 

Similary, the greatest lower bound of $f(K)$ is attained by some member of $K$. □

**Defn.** A function $f$ is called **uniformly continuous** if for each $\epsilon > 0$, $\exists \delta > 0$ such that whenever $x_1, x_2 \in \text{dom}(f)$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$.

**Corollary.** Each continuous function on $[a,b]$ is uniformly continuous.
*Proof. Suppose not, then negating the definition implies that there exist an $\epsilon_0 > 0$ such that for each $n \in \mathbb{N}$ we can find $x_n, y_n \in K$ with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$. $K$ is compact so we can find a subsequence $\{x_{n_k}\}^\infty_{k=1}$ of $\{x_n\}^\infty_{n=1}$ which converges to some $x_0$ belonging to $K$. Notice that $\{y_{n_k}\}^\infty_{k=1}$ also converges to $x_0$ (use an $\epsilon/2$ proof). But $f$ is continuous at $x_0$, so

$$\epsilon_0 \leq |f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(x_0)| + |f(x_0) - f(y_{n_k})| \rightarrow 0 \text{ as } k \rightarrow \infty$$

which is a contradiction. □