1. Give an example of each of the following and (very) briefly justify your answer:

(a) A bounded set of real numbers that is not compact.

\[(0,1] \text{ is bounded but not closed} \implies \text{ not compact.}\]

(b) A connected set of real numbers that is not compact.

\[(0,1] \text{ is an interval} \implies \text{ connected, but again not compact.}\]

(c) A real-valued continuous function that does not satisfy the Extreme Value Theorem.

\[f(x) = \begin{cases} x, & 0 < x \\ -x, & 0 \leq x < 1 \end{cases}\]

\[\text{or} \quad f(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}\]

(d) A real-valued continuous function that does not satisfy the Intermediate Value Theorem.

\[f(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}\]

\[\text{then} \quad f \text{ is continuous on its domain but not bounded.}\]

2. a. State the Bolzano-Weierstrass property for a subset of real numbers. \(K\).

Each infinite subset \(A\) of \(K\) has a limit point which belongs to \(K\).

b. State the sequential compactness property for a subset of real numbers. \(K\).

Each sequence in \(K\) has a subsequence which converges to a member of \(K\).
3. a) Define open cover for a set.

A cover \( C \) is an open cover for a set \( K \) if \( C \) is a collection of open sets and \( K = \bigcup C \).

b) Define what it means for a set to be compact.

A set \( K \) is compact if each open cover of \( K \) has a finite subcover which covers \( K \).

c) Suppose \( K \) is compact and \( f: K \rightarrow \mathbb{R} \) is continuous. Prove that \( f[K] \) is compact.

Let \( C \) be an open cover of \( f[K] \). Define \( C' := \{ f^{-1}(U) | U \in C \} \).

Each \( U \in C \) is open since \( \exists \, V \) open \( \Rightarrow \exists \, f(V) \) open (by \( f \) is continuous).

Since \( C \) is a cover for \( C' \), then \( C' \) is a cover for \( K \). \( K \) is compact, \( C' \) is an open cover for \( K \), so has a finite subcover \( \{ O_1, O_2, \ldots, O_n \} \).

But then \( O_1 \supseteq O_i = f^{-1}(O_i) \) for \( 1 \leq i \leq n \) forms a finite subcover of \( C \) which covers \( f[K] \). \( \therefore f[K] \) is compact.

4. State and sketch a proof of the Heine-Borel theorem.

Either use the proof in the course notes or if you prefer to use a proof indicated in the reference text, then you must fill in some details as follows:

Thus if \( a < b \), then the interval \([a, b]\) is compact.

The proof given in class is replicated in the notes posted on the web.

The proof below is one (with completed details) that a good portion of the class wrote.

A = \[ a, x \] \( \subseteq \mathbb{R} \) has a finite subcover from \( C \). Since \( a \leq A \), then \( A \) is nonempty. If \( A \) is not bounded, then \([a, b]\) has a finite subcover from \( C \) since \( A \) is an interval \( \setminus \) in this case \( A = [a, \infty) \).

If \( A \) is bounded, then let \( \gamma := \sup A \). If \( \gamma > b \), then done since \( b \notin A \).

If \( \gamma \leq b \), then \( C \) an open cover for \([a, b]\) \Rightarrow \exists \, O_0 \supseteq \gamma \in C \). If \( O_0 \) open \( \Rightarrow \exists \, \varepsilon > 0 \) \( \Rightarrow N_{\varepsilon}(\gamma) \subseteq O_0 \). Since \( \gamma \) is the least upper bound of \( A \), \( \exists \, \varepsilon > 0 \) \( \exists \, x \in A \) such that \( \gamma - \varepsilon < x \leq \gamma \).

Since \( x \in A \), \( \exists \, \text{ a finite subcover } O_1, \ldots, O_n \in C \) covering \([a, x]\).

But \( \bigcup_{j=0}^{n} O_j = [a, \gamma + \varepsilon] \Rightarrow \gamma + \varepsilon / 2 \in A \ast \gamma = \sup A \). Therefore \( \gamma \leq b \) is impossible.
5. Prove that each closed and bounded subset of real numbers is a compact set.

Suppose \( C \) is closed and bounded. Since \( C \) is bounded \( \exists M \in \mathbb{R} \) \( \exists \ |x| \leq M, \forall x \in C \). Hence \( C \subseteq [-M, M] =: K \). By the Heine-Borel theorem \( K \) is compact. But closed subsets of compact sets are also compact, so \( C \) is compact.

6. a. Define “disconnection” for a set \( A \).

A disconnection \((A_1, A_2)\) for a set \( A \) satisfies

\[
A = A_1 \cup A_2 \quad \text{and} \quad A_1 \cap A_2 = \emptyset
\]

with \( A_1, A_2 \) open relative to \( A \).

b. Define what it means for a set \( A \) to be a connected subset of real numbers.

\( A \) is connected means \( A \) has no disconnections.

c. Show that a subset \( A \) of real numbers is connected implies that \( A \) is an interval.

If \( A \) is connected, but not an interval, then \( \exists a_1, a_2 \in A \) and \( a \notin A \) with \( a_1 < a < a_2 \). Define

\[
A_1 = (-\infty, a) \cap A, \\
A_2 = (a, \infty) \cap A,
\]

then \( a_1 \in A_1, a_2 \in A_2 \), \( A_1 \cap A_2 = \emptyset \), \( A = A_1 \cup A_2 \).

Hence \( A \) is not connected. ※
7. State and prove the Intermediate Value Theorem

**Theorem** Suppose \( f: [a, b] \to \mathbb{R} \) is continuous and that \( x_1, x_2 \in [a, b] \). If \( \eta \) is between \( f(x_1) \) and \( f(x_2) \), then \( \exists \xi \) between \( x_1 \) and \( x_2 \) so that \( \eta = f(\xi) \).

**Proof** WLOG \( x_1 \leq x_2 \) and set \( I = [x_1, x_2] \). Since \( I \) is connected \( f \) is continuous, then the image \( f[I] \) is connected. But \( f[I] \subseteq \mathbb{R} \) so \( f[I] \) is an interval \( J \). By hypothesis, \( \eta \) is between \( f(x_1) \) and \( f(x_2) \), both of which belong to the interval \( J \). Hence \( \eta \in f[I] \). This just means \( \exists \xi \in I \ni f(\xi) = \eta \). \( \blacksquare \)