Math 554 – Differentiation
Handout #8

Defn. A function f is said to be *differentiable at* x_0 if

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

exists. In this case the limit is called the *derivative* of f at x_0 and is denoted $f'(x_0)$.

Note.

1. This definition is equivalent to the requirement that the following limit exist:

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0).$$

2. This, in turn, is equivalent to the following statement about how fast $f(x)$ converges to $f(x_0)$ as $x \to x_0$:

   there exists a function η such that $\lim_{x\to x_0} \eta(x) = 0$ and

$$(*) \qquad f(x)-f(x_0) = (x-x_0)\left(f'(x_0)+\eta(x)\right).$$

   Note that if we define $\eta(x_0)=0$, then without loss of generality we can assume that η is continuous at x_0.

Examples:

1. If $f(x) := x^2$, then $f'(x) = 2x$.
2. If $g(x) := |x|$, then $g'(0)$ does not exist.
3. If $h(x) := x|x|$, then $h'(x)$ exists and equals $2|x|$.

Theorem. If f is differentiable at x_0, then f is continuous at x_0.
Proof. Use (*) and let $x \to x_0$. □

Theorem. (Basic rules of differentiation: sums, products, quotients) Suppose that f and g are differentiable at x_0, then

1. $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.
2. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.
3. $(f/g)'(x_0) = \left(g(x_0)f'(x_0) - f(x_0)g'(x_0)\right)/g(x_0)^2, \quad$ if $g(x_0) \neq 0$.

Theorem. (Chain rule) If f is differentiable at x_0 and g is differentiable at $y_0 := f(x_0)$, then $h := g \circ f$ is differentiable at x_0 and

$$h'(x_0) = g'(f(x_0))\, f'(x_0)$$

Proof. Use (*) for f at x_0 and for g at $y_0 := f(x_0)$:

$$\begin{aligned}\frac{h(x)-h(x_0)}{x-x_0} &= \frac{g(y)-g(y_0)}{x-x_0} = \frac{y-y_0}{x-x_0}(g'(y_0)+\eta_2(y)) \\ &= \frac{f(x)-f(x_0)}{x-x_0}(g'(y_0)+\eta_2(y)) \\ &= (f'(x_0)+\eta_1(x))(g'(y_0)+\eta_2(y))\end{aligned}$$

where $y := f(x)$. The proof is completed by using this equation, letting $x_n \to x_0$, and noticing that $y_n \to y_0$ where $y_n := f(x_n)$. □

Theorem. (Rolle's Theorem) Suppose that ϕ is differentiable on (a, b), is continuous on $[a, b]$, and vanishes at the endpoints, then there exists x_0 strictly between a and b such that $\phi'(x_0) = 0$.
Proof. If ϕ is constant, then any point can be selected for x_0. Otherwise, we may assume WLOG that ϕ has positive values. By the Extreme Value Theorem, let x_0 be such that $\phi(x) \leq \phi(x_0)$ for all $a \leq x \leq b$. First, let $x_n \downarrow x_0$, then since x_0 gives a max, we have

$$0 \geq \frac{\phi(x_n)-\phi(x_0)}{x_n - x_0} \to \phi'(x_0)$$

and so, by the Squeeze Theorem, $\phi'(x_0) \leq 0$. Similarly, $\phi'(x_0) \geq 0$. □

Note. Within the proof we actually established the critical point procedure of calculus: local max and min can only occur at critical points.

Corollary. Suppose that f is a differentiable function on (a, b) and is continuous on $[a, b]$. Then f' vanishes identically if and only if f is a constant function.

Corollary. (Mean Value Theorem) Suppose that f is differentiable on (a, b) and is continuous on $[a, b]$, then there exists x_0 strictly between a and b such that

$$f'(x_0) = \frac{f(b)-f(a)}{b-a}.$$

Proof. Let

$$\phi(x) := f(x) - \left[\tfrac{f(b)-f(a)}{b-a}(x-a) + f(a)\right]$$

and apply Rolle's theorem. □

Defn. F is called an *anti-derivative* of f if F is differentiable and $F'(x) = f(x)$

Corollary. If both F and G are anti-derivatives of f, then they differ by a constant, i.e. there exists a constant c such that $F(x) - G(x) = c$, for all $x \in dom(f)$.