Defn 1. A function $f$ is called Lipschitz if there is an $M > 0$ such that

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2|, \text{ for all } x_1, x_2 \in \text{dom}(f).$$

If $M < 1$, then $f$ is called a contraction.

Theorem 1. Each Lipschitz function is uniformly continuous.

Theorem 2. Suppose that $K$ is compact and $f : K \to K$ is a contraction, then $f$ has a fixed point in $K$.

Proof Let $x_0$ be an arbitrary point in $K$. Define inductively,

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots$$

We claim that the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent to some $\alpha \in K$. First note that for each $n \in \mathbb{N}$

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq M |x_n - x_{n-1}|.$$

Hence, by induction, for each $n \in \mathbb{N}$

$$|x_{n+1} - x_n| \leq M^n |x_1 - x_0|.$$

We then see that if $m > n$, then $m = n + k$ where $k \in \mathbb{N}$ and

$$|x_{n+k} - x_n| \leq |x_{n+k} - x_{n+k-1}| + |x_{n+k-1} - x_{n+k-2}| + \ldots + |x_{n+1} - x_n|$$

$$\leq (M^{n+k-1} + M^{n+k-2} + \ldots + M^n)|x_1 - x_0|$$

$$= M^n(1 + M + \ldots + M^{k-1})|x_1 - x_0|$$

$$\leq \frac{|x_1 - x_0|}{1-M} M^n$$

and so $\{x_n\}_{n=1}^{\infty}$ is Cauchy. It must converge to some limit $\alpha$ which will belong to $K$ since $K$ is closed. But $f$ is continuous, so $x_{n+1} = f(x_n) \to f(\alpha)$. Notice also that $x_{n+1} \to \alpha$, so $\alpha$ is our fixed point. $\square$

Theorem 3. Suppose that $f : [a, b] \to K$ is one-to-one, onto and continuous, then $f^{-1}$ is continuous.

Proof (#1) Suppose that $g := f^{-1}$ and $y_n \to y_0 \in K$. There exists unique $x_n \in [a, b]$ such that $f(x_n) = y_n$, or equivalently, $x_n = g(y_n)$. If $x_n \not\to x_0$, then there exists $\epsilon_0 > 0$ and a subsequence $x_{n_k}$ such that $|x_{n_k} - x_0| \geq \epsilon_0$. This sequence in turn has
a subsequence which converges in $K$ to some $z \in K$. We may as well assume that the subsequence is the sequence $\{x_{n_k}\}$. $f$ is continuous so $y_{n_k} = f(x_{n_k}) \to f(z)$. But then $f(z) = y_0 = f(x_0)$. $f$ is one-to-one, so $z = x_0$, which is a contradiction, since $|x_{n_k} - x_0| \geq \epsilon_0$. □

Proof (#2) Let $O \subseteq [a, b]$ be relatively open, then $(f^{-1})^{-1}(O) = f(O)$. Let $C$ be the complement in $[a, b]$ of $O$, then $C$ is closed and hence compact. Therefore $f(C)$ is compact in $K$ and consequently it is closed. Its complement in $K$ must then be open. That complement however is $f(O)$. □