

**Compactness**

Handout #7

**Defn 1.** Suppose that \( K \subseteq \mathbb{R} \). A collection \( G \) of open subsets such that

\[
K \subseteq \bigcup_{O \in G} O.
\]

is called an open cover of \( K \). \( K \) has a finite subcover from \( G \) if there exist \( O_1, O_2, \ldots, O_n \) in \( G \) for which

\[
K \subseteq \bigcup_{j=1}^n O_j.
\]

**Defn 2.** \( K \) is called compact, if each open cover \( G \) of \( K \) has a finite subcover.

**Theorem 1.** The continuous image of a compact set is compact.

*Proof.* Suppose \( f : K \to \mathbb{R} \) is continuous and \( K \) is compact. Each open cover \( C \) of \( f[K] \) can be drawn back to an open cover \( \tilde{C} \) of \( K \), by considering the sets

\[
\tilde{O} := f^{-1}(O), \; O \in C.
\]

\( K \) compact implies that we may draw a finite subcover from \( \tilde{C} \). Each of these members is the inverse image (under \( f \)) from a member of \( C \). These form the desired subcover of \( f[K] \). \( \Box \)

**Theorem 2.** (Heine-Borel) Suppose that \( a \leq b \), then the interval \([a, b] \) is compact.

*Proof.* Let \( C \) be an open cover for \([a, b] \) and consider the set

\[
A := \{ a \leq x \leq b + 1 \mid [a, x] \text{ has a finite open cover from } C \}.
\]

Note that \( A \) is bounded and nonempty (since \( a \in A \)). Let \( \gamma := \text{lub}(A) \). It is enough to show that \( \gamma > b \), since if \( x_1 \in A \) and \( a \leq x \leq x_1 \), then \( x \in A \). Suppose instead that \( \gamma \leq b \), then there must be some \( O_0 \in C \) such that \( \gamma \in O_0 \). But \( O_0 \) is open, so there exists \( \delta > 0 \) so that \( B_\delta(\gamma) \subseteq O_0 \). Since \( \gamma \) is the least upper bound for \( A \), then there is an \( x \in A \) such that \( \gamma - \delta < x \leq \gamma \). But \( x \in A \) implies there are members \( O_1, \ldots, O_n \) of \( C \) whose union covers \([a, x] \). The collection \( O_0, O_1, \ldots, O_n \) covers \([a, \gamma + \delta/2] \). Contradiction, since \( \gamma \) is the least upper bound for the set \( A \). \( \Box \)

**Theorem 3.** Each closed subset \( C \) of a compact set \( K \) is compact.

*Proof.* Let \( \tilde{G} \) be an open cover for \( C \). Let \( \mathcal{O}_0 \) be the complement of \( C \), then \( \mathcal{O}_0 \) is open and \( \mathcal{G} := \tilde{G} \cup \{ \mathcal{O}_0 \} \) is an open cover for \( K \). There is a finite subcover of
which covers $K$ and hence $C$. This subcover (dropping $O_0$ if it appears) is the desired finite subcover for $C$.  \[\square\]

**Defn 3.** Suppose $\{a_n\}$ is a sequence. A sequence $\{b_k\}$ is called a subsequence of $\{a_n\}$ if there exists a strictly increasing sequence of natural numbers

$$n_1 < n_2 < \ldots < n_k < \ldots$$

such that $b_k = a_{n_k}$, $k = 1, 2, \ldots$

**Theorem 4.** Suppose that $K \subseteq \mathbb{R}$, then TFAE:

a.) $K$ is compact,

b.) $K$ is closed and bounded,

c.) each sequence in $K$ has a subsequence which converges to a member of $K$,

d.) (Bolzano-Weierstrass) each infinite subset of $K$ has a limit point in $K$.

**Proof.**

**(a) \Rightarrow (b):** To show that $K$ is bounded, consider the open cover of $K$ consisting of the collection of nested open intervals $O_n := (-n, n)$, $n \in \mathbb{N}$. To show that $K$ is closed, let $x_0$ be a limit point of $K$. Assume to the contrary that $x_0 \notin K$. Consider the open cover of $K$ consisting of the collection of nested open sets $O_n := \{x \in \mathbb{R} | |x - x_0| > 1/n\}$, $n \in \mathbb{N}$. Any finite subcollection which would cover $K$ would have union whose complement would be a neighborhood of $x_0$ not intersecting $K$. This shows that $x_0$ could not be a limit point of $K$.

**(b) \Rightarrow (d):** We use the ‘divide and conquer’ method, better known as the ‘bisection’ method. Let $A$ be an infinite subset of $K$. Since $K$ is bounded, there is an interval $[a, b]$ such that $K \subseteq [a, b]$. Inductively define the closed subintervals as follows. Let $[a_0, b_0] := [a, b]$. Either the left or right half of $[a_0, b_0]$ contains an infinite number of members of $K$. In the case that it is the right half, set $[a_1, b_1] := [(b_0 + a_0)/2, b_0]$. Set $[a_1, b_1]$ equal to the left half of $[a_0, b_0]$ otherwise. Inductively, let $[a_{n+1}, b_{n+1}]$ be the half of $[a_n, b_n]$ which contains an infinite number of members of $A$. Notice that the length of this interval is $(b - a)/2^{n+1}$, that the $a_n$’s satisfy $a_n \leq a_{n+1} \leq \ldots < b$ and so must converge to some real number $a \leq x_0 \leq b$. Each neighborhood of $x_0$ will contain one of the intervals $[a_n, b_n]$ and hence will contain an infinite number of members of $A$, i.e. $x_0$ is a limit point of $A$. This also shows that $x_0$ is a limit point of the closed set $K$ and must therefore belong to $K$.

**(d) \Rightarrow (c):** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $K$. If the sequence’s image is finite, then we may construct a constant subsequence which has the value which we may choose as any of the values of $\{x_n\}_{n=1}^{\infty}$ which is repeated infinitely often. Otherwise, let $A$
be the range of the sequence. Then $A$ is an infinite subset of $K$. By the Bolzano-Weierstrass property, $A$ must have a limit point ($x_0$ say) which belongs to $K$. For each $k \in \mathbb{N}$, we may find an integer $n_k$ larger than those previously picked (i.e., $n_1, \ldots, n_{k-1}$), so that $|x_{n_k} - x_0| < 1/k$. This is the desired subsequence.

$(c) \Rightarrow (b)$: If $K$ were not bounded, then there would exist a sequence $x_n \in K$ such that $|x_n| > n$. If this sequence had a subsequence which converged, then it would have to be bounded. But each subsequence of $\{x_n\}$ is clearly unbounded. To show that $K$ is closed, we let $x_0$ be a limit point of $K$ which is not in $K$. We can then find a sequence $\{x_n\}$ from $K$ which converges to $x_0$. By condition (c), this has to have a subsequence which converges to a member of $K$. Contradiction. Each subsequence of a convergent sequence converges to the same limit, in this case $x_0$, which does not belong to $K$. □

**Corollary 1.** Each continuous function $f$ on a compact set $K$ is bounded.

*Proof.* The set $f(K)$ is compact and is therefore bounded. □

**Corollary 2.** (Extreme Value Theorem) Each continuous function on a compact set $K$ attains its maximum (resp. minimum).

*Proof.* The set $f(K)$ is compact and is therefore bounded and closed. Hence the least upper bound $\gamma$ for $f(K)$ must belong to $f(K)$. Therefore, there is an $x_0 \in K$ such that $\gamma = f(x_0)$ and so

$$f(x) \leq f(x_0), \text{ for all } x \in K.$$ 

Similarly, the greatest lower bound of $f(K)$ is attained by some member of $K$. □

**Defn 4.** A function $f$ is called uniformly continuous if for each $\epsilon > 0$, $\exists \delta > 0$ such that whenever $x_1, x_2 \in \text{dom}(f)$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$.

**Corollary 3.** Each continuous function on $[a, b]$ is uniformly continuous.

*Proof.* Suppose not, then negating the definition implies that there exist an $\epsilon_0 > 0$ such that for each $n \in \mathbb{N}$ we can find $x_n, y_n \in K$ with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$. $K$ is compact so we can find a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which converges to some $x_0$ belonging to $K$. Notice that $\{y_{n_k}\}_{k=1}^{\infty}$ also converges to $x_0$ (use an $\epsilon/2$ proof). But $f$ is continuous at $x_0$, so

$$\epsilon_0 \leq |f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(x_0)| + |f(x_0) - f(y_{n_k})| \to 0 \text{ as } k \to \infty$$

which is a contradiction. □