We need to recall the following:

**Earlier Definitions**

- A point $x_0$ is called a *limit point* of a set $A$ if each nbhd of $x_0$ contains a member of $A$ different from $x_0$, i.e. for each $\epsilon > 0$, $(N_\epsilon(x_0) \setminus \{x_0\}) \cap A \neq \emptyset$.
- A point $x_0 \in A$ is called an *isolated point* of $A$ if $x_0$ belongs to $A$ but is not a limit point.

**Earlier Theorems**

- A set $F$ is closed if and only if it contains all its limit points.
- $x_0$ is a limit point of a set $A$ if and only if there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \to x_0$, but $x_n \neq x_0$, ($\forall n \in \mathbb{N}$).

With these ideas we begin the study continuity of functions, which is very naturally framed in a general metric space.

**Defn.** Suppose that $x_0$ is a limit point of the domain of a function $f : A \to B$, then $f$ is said to have a *limit $L$ as $x$ approaches $x_0$* if,

$$\forall \epsilon > 0, \exists \delta > 0 \ni (x \in \text{dom}(f) \& 0 < d_A(x, x_0) < \delta) \implies d_B(f(x), L) < \epsilon.$$  

In this case, we use the notation,

$$\lim_{x \to x_0} f(x) = L.$$  

**Defn.** Suppose $f : A \to B$, metric spaces. If $x_0 \in A$, then $f$ is said to be *continuous at $x_0$* if for each $\epsilon > 0$ there is a $\delta > 0$ so that if $x \in A$ and $d_A(x, x_0) < \delta$, then $d_B(f(x), f(x_0)) < \epsilon$.

**Note.** Remember that a point $x_0 \in A$ is either an isolated point of $A$ or it is a limit point of $A$. Considering these two cases separately, the definition for continuity of $f$ at a point can be seen to be equivalent to the following in each of the respective situations:

1. if $x_0$ is an isolated point of $A$, then $f$ is automatically continuous.
2. if $x_0$ is a limit point of the domain $A$, then the condition $\lim_{x \to x_0} f(x) = f(x_0)$ must hold.

**Defn.** Consider a set $B^* \subseteq B$. A set $\tilde{O} \subseteq B^*$ is called \textit{open relative to $B^*$} (or briefly, \textit{relatively open}) if $\tilde{O} = O \cap B^*$ for some open set $O \subseteq B$. That is to say, $\tilde{O}$ is just the restriction to $B^*$ of an open set in the whole space.

**Theorem.** Let $f: A \to B$, where $A, B$ are metric spaces, then TFAE

a.) $f$ is continuous at each point of its domain,

b.) for each limit point $x_0$ of the domain $A$, $\lim_{x \to x_0} f(x)$ exists & equals $f(x_0)$.

c.) for each sequence $x_n \to x_0$, then $f(x_n) \to f(x_0)$ must hold,

d.) $f^{-1}[O]$ is open for each open subset $O$ of $B$.

We will be concentrating on the metric space of real numbers.

**Note.** The same proof above for continuity at a point $x_0$ can be used to show the corresponding result for limits holds. The only difference is that $f$ is not required to be defined at $x_0$. The statement reads as:

\textit{Suppose that } $f: A \to B$ \textit{is a real-valued function of a real variable, i.e. } $A, B \subseteq \mathbb{R}$. \textit{If } $x_0$ \textit{is a limit point of the domain of } $f$, \textit{then TFAE}:

a.) $\lim_{x \to x_0} f(x) = L$,

b.) For every sequence $\{x_n\}$ in the domain of $f$, if $x_n \to x_0$, then $f(x_n) \to L$.

**Corollary.** The finite sum, product, or the quotient of continuous functions is each continuous on their respective domains.

**Corollary.** All polynomials are continuous. Rational functions are continuous on their domains.

**Theorem.** The composition of continuous functions is continuous.

**Example.** Each of the following are examples of continuous functions on their respective domains:

1. $f(x) := |x|$
2. $g(x) := \sqrt{x}$
3. $F(x) := \sqrt{\frac{x^2 - 2x + 5}{x^3 - 1}}$
1. Suppose that $x_0$ is a limit point of the domain of $f + g$ and that both $f$ and $g$ have limits at $x_0$, then prove that

$$\lim_{x \to x_0} (f + g)(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x).$$

2. Suppose that $f$ is defined by

$$f(x) := \begin{cases} 
3x + 2, & \text{if } -1 \leq x \\
-2x + 1, & \text{if } x < -1.
\end{cases}$$

Determine at each point whether or not $f$ is continuous. Justify your answer.

3. Determine the domain of $F(x) := \sqrt{\frac{x^2 - 2x + 5}{x^3 - 1}}$ and carefully show that $F$ is continuous at each point of its domain.