Name: ___________________________ 4 Digit CODE: ______

Directions: To receive credit, you must justify your statements unless otherwise stated. Answers should be provided in complete sentences.

1. a.) Define metric.

E is a set \( d : E \times E \rightarrow \mathbb{R}^+ \) with the properties:

1. \( d(p, q) \geq 0 \), all \( p, q \in E \)
2. \( d(p, q) = 0 \iff p = q \).
3. \( d(p, q) = d(q, p) \), all \( p, q \in E \).
4. \( d(p, r) \leq d(p, q) + d(q, r) \), all \( p, q, r \in E \).

b.) Give two examples of metric spaces. (You do not need to verify the properties.)

Both the set and the specific metric must be provided.

Notes or Text:

2. Let \((E, d)\) be a metric space.

a.) Define open set.

\( S \) is open means if \( p \in S \) then there exists \( \varepsilon > 0 \) so that \( B_\varepsilon(p) \subseteq S \).

Notes or Text:

b.) Prove that an open ball is an open set.

Suppose \( p \in B_\varepsilon(p_0) \). Let \( \varepsilon = r - d(p, p_0) > 0 \). If \( q \in B_\varepsilon(p) \), then \( d(p, q) < \varepsilon \) and so

\[
d(q, p_0) \leq d(q, p) + d(p, p_0) < \varepsilon + d(p, p_0) = r.
\]

Therefore, \( B_\varepsilon(q) \subseteq B_r(p_0) \), \( B_r(p_0) \) is open. \( \Box \)

c.) Let \( d \) be the discrete metric on a set \( E \). Prove that each subset \( S \) of \( E \) is a closed set.

It suffices to show each \( S \) is open, by considering complements.

But each point is open, since \( \{ p_0 \} = B_{\varepsilon}(p_0) \) is open. Arbitrary unions of open sets are open, so every set in \((E, d)\) is open. \( \Box \)
3. a.) Give the definition of a closed set.

Notes or text: C is closed means the complement of C is open.

b.) Give the definition of a limit point of a set.

Notes or text: p is a limit point of S means each open ball of p contains a point of S different from p.

c.) Prove that a set is closed if and only if it contains all its limit points.

Let \( S' \) be the set of all limit points of \( S \). Let \( O := \mathcal{G} S \).

\( \Rightarrow \) Suppose \( S \) is closed \( \Rightarrow \) let \( p_o \in S' \). By definition \( O = \mathcal{G} S \) is open.

If \( p_o \) is not in \( S \) (i.e. \( p_o \not\in O \)), then \( \exists B_\varepsilon(p_o) \) which misses \( S \). \( \therefore \)

\( p_o \) is a limit point of \( S \). Hence \( p_o \) must belong to \( S \) and \( S' \subseteq S \).

\( \Leftarrow \) It is enough to show \( O \) is open. Suppose not, then \( \exists p_o \in O \)

such that \( \forall \varepsilon > 0 \ B_\varepsilon(p_o) \not\subseteq O \). That is, \( \forall \varepsilon > 0 \) there is a member of \( S_0 = S \) which belongs to \( B_\varepsilon(p_o) \). That member cannot be \( p_o \) because \( p_o \in O \). Hence \( p_o \) is a limit point of \( S \). But \( S' \subseteq S \), so \( p_o \in S \). \( \therefore \)

4. Using the definition of "convergence of a sequence," prove that

a.) \( \{a_n\} \) converges to \( a \) implies that \( a_n^2 \) converges to \( a^2 \).

\( \sum a_n \) convergent implies \( \{a_n\} \) is bounded, so \( \exists M \ni\)

\[ |a_n| \leq M, \quad n = 1, 2, \ldots \]

Let \( \varepsilon > 0 \), then

(\( \star \)) \[ |a_n^2 - a^2| = |a_n - a||a_n + a| \leq |a_n - a|(|a_n| + |a|) \leq (M + |a|)|a_n - a|, \]

\( a_n \to a \) \( \Rightarrow \) \( \exists N \ni |a_n - a| < \varepsilon \) if \( n \geq N \). Hence

b.) \( \{a_n\} \) converges to \( a \) implies that \( |a_n| \) converges to \( |a| \).

Let \( \varepsilon > 0 \). Since \( \lim_{n \to \infty} a_n = a \), then \( \exists N \ni\)

\[ n \geq N \Rightarrow |a_n - a| < \varepsilon. \]

By the reverse triangle inequality

\[ |a_n| - |a| \leq |a_n - a| < \varepsilon \]

if \( n \geq N \). Hence \( \lim_{n \to \infty} |a_n| = |a|. \)
5. Using the **properties** of limits, determine whether or not the following limit exists. Be sure to state which property you are using as you show your work.

a.) \( a_n = 1 - \frac{2}{n} \)

We know that \( a_n = \frac{1}{n} \to 0 \) as \( n \to \infty \) \( \frac{1}{n} \) \( \to 0 \) as \( n \to \infty \), so \( c_n = \frac{1}{n} - \frac{2}{n} \to 0 \).

\( d_n = 1 \to 1 \) as \( n \to \infty \), so \( 1 - \frac{2}{n} = d_n + c_n \to 1 - 0 = 1 \).

(For this problem an \( \varepsilon \)-proof is more direct & easier.)

b.) \( b_n = 2 + \frac{3}{n^2} \)

Similar to part a, \( \frac{1}{n} \to 0 \) \( \Rightarrow \frac{1}{n} - (\frac{1}{n^2}(\frac{1}{n^2}) \to 0 \).

Therefore by using the \( \varepsilon \)-proof, the limit of \( b_n \) is \( 2 + \frac{3}{n^2} \to 2 + 0 \cdot 0 = 2 \).

Typo corrected during test.

c.) Consider the sequence, \( c_n = \frac{n-1}{2n^2 + 3} \). Use parts a.) and b.) to determine the convergence of \( \{c_n\} \).

\[ c_n = \frac{n-1}{2n^2 + 3} = \frac{1}{n} \cdot \frac{1-\frac{1}{n}}{2+\frac{3}{n^2}} = \frac{1}{n} \cdot \frac{\frac{1}{n}}{b_n} \to 0 \cdot \frac{1}{2} = 0, \] since \( b_n \to 2 \neq 0 \).

Using quotients & products of limits.

6. Suppose that \( E \) is a metric space and \( S \subset E \) is complete. Prove that \( S \) is closed.

Suppose \( S \) is not closed, then there exists a limit point \( p \) of \( S \) which is not in \( S \). By our previous work, we know that there exists a sequence \( \{p_n\} \subset S \) such that \( \lim_{n \to \infty} p_n = p \) and \( p \notin S \) but \( p \notin S \). Convergent sequences are Cauchy, so \( \{p_n\} \subset S \) \( S \) is Cauchy. \( S \) is complete so \( \{p_n\} \subset S \) is convergent to a limit in \( S \). Hence \( p \in S \). Therefore \( S \) must be closed.