Defn. A collection of \( n + 1 \) distinct points of the interval \([a, b]\)

\[
P := \{x_0 = a < x_1 < \cdots < x_{i-1} < x_i < \cdots < b =: x_n\}
\]

is called a partition of the interval. In this case, we define the norm of the partition by

\[
\|P\| := \max_{1 \leq i \leq n} \Delta x_i,
\]

where \( \Delta x_i := x_i - x_{i-1} \) is the length of the \( i \)-th subinterval \([x_{i-1}, x_i]\).

Defn. For a given partition \( P \), we define the Riemann upper sum of a function \( f \) by

\[
U(f, P) := \sum_{i=1}^{n} M_i \Delta x_i
\]

where \( M_i \) denotes the supremum of \( f \) over each of the subintervals \([x_{i-1}, x_i]\). Similarly, we define the Riemann lower sum of a function \( f \) by

\[
L(f, P) := \sum_{i=1}^{n} m_i \Delta x_i
\]

where \( m_i \) denotes the infimum of \( f \) over each of the subintervals \([x_{i-1}, x_i]\). Since \( m_i \leq M_i \), we note that

\[
L(f, P) \leq U(f, P).
\]

for any partition \( P \).

Defn. Suppose \( P_1, P_2 \) are both partitions of \([a, b]\), then \( P_2 \) is called a refinement of \( P_1 \), denoted by

\[
P_1 \prec P_2,
\]

if as sets \( P_1 \subseteq P_2 \).

Note. If \( P_1 \prec P_2 \), it follows that \( \|P_2\| \leq \|P_1\| \) since each of the subintervals formed by \( P_2 \) is contained in a subinterval arising from \( P_1 \).

Lemma. If \( P_1 \prec P_2 \), then

\[
L(f, P_1) \leq L(f, P_2).
\]

and

\[
U(f, P_2) \leq U(f, P_1).
\]

Proof. Suppose first that \( P_1 \) is a partition of \([a, b]\) and that \( P_2 \) is the partition obtained from \( P_1 \) by adding an additional point \( z \). The general case follows by induction, adding one point at a time. In particular, we let

\[
P_1 := \{x_0 = a < x_1 < \cdots < x_{i-1} < x_i < \cdots < b =: x_n\}
\]

and

\[
P_2 := \{x_0 = a < x_1 < \cdots < x_{i-1} < z < x_i < \cdots < b =: x_n\}
\]
for some fixed \( i \). We focus on the upper Riemann sum for these two partitions, noting that the inequality for the lower sums follows similarly. Observe that

\[
U(f, P_1) := \sum_{j=1}^{n} M_j \Delta x_j
\]

and

\[
U(f, P_2) := \sum_{j=1}^{i-1} M_j \Delta x_j + M(z - x_{i-1}) + \tilde{M}(x_i - z) + \sum_{j=i+1}^{n} M_j \Delta x_j
\]

where \( M := \sup_{[x_{i-1}, x_i]} f(x) \) and \( \tilde{M} := \sup_{[z, x_i]} f(x) \). It then follows that \( U(f, P_2) \leq U(f, P_1) \) since

\[ M, \tilde{M} \leq M_i. \]

**Defn.** If \( P_1 \) and \( P_2 \) are arbitrary partitions of \([a, b]\), then the *common refinement* of \( P_1 \) and \( P_2 \) is defined as the formal union of the two.

**Corollary.** Suppose \( P_1 \) and \( P_2 \) are arbitrary partitions of \([a, b]\), then

\[ L(f, P_1) \leq U(f, P_2). \]

**Proof.** Let \( P \) be the common refinement of \( P_1 \) and \( P_2 \), then

\[ L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2). \]

**Defn.** The *lower Riemann integral* of \( f \) over \([a, b]\) is defined to be

\[
\int_{a}^{b} f(x) \, dx := \sup_{\text{all partitions } P \text{ of } [a,b]} L(f, P).
\]

Similarly, the *upper Riemann integral* of \( f \) over \([a, b]\) is defined to be

\[
\int_{a}^{b} f(x) \, dx := \inf_{\text{all partitions } P \text{ of } [a,b]} U(f, P).
\]

By the definitions of least upper bound and greatest lower bound, it is evident that for any function \( f \) there holds

\[
\int_{a}^{b} f(x) \, dx \leq \int_{a}^{b} f(x) \, dx.
\]

**Defn.** A function \( f \) is *Riemann integrable over \([a, b]\)* if the upper and lower Riemann integrals coincide. We denote this common value by \( \int_{a}^{b} f(x) \, dx \).

**Theorem.** A necessary and sufficient condition for \( f \) to be Riemann integrable is given \( \epsilon > 0 \), there exists a partition \( P \) of \([a, b]\) such that

\[ (*) \quad U(f, P) - L(f, P) < \epsilon. \]
Proof. First we show that (*) is a sufficient condition. This follows immediately, since for each $\epsilon > 0$ that there is a partition $P$ such that (*) holds,

$$
\int_a^b f(x) \, dx - \int_a^b f(x) \, dx \leq U(f, P) - L(f, P) < \epsilon.
$$

Since $\epsilon > 0$ was arbitrary, then the upper and lower Riemann integrals of $f$ must coincide.

To prove that (*) is a necessary condition for $f$ to be Riemann integrable, we let $\epsilon > 0$. By the definition of the upper Riemann integral as a infimum of upper sums, we can find a partition $P_1$ of $[a, b]$ such that

$$
\int_a^b f(x) \, dx \leq U(f, P_1) < \int_a^b f(x) \, dx + \epsilon/2
$$

Similarly, we have

$$
\int_a^b f(x) \, dx - \epsilon/2 < L(f, P_2) \leq \int_a^b f(x) \, dx.
$$

Let $P$ be a common refinement of $P_1$ and $P_2$, then subtracting the two previous inequalities implies,

$$
U(f, P) - L(f, P) \leq U(f, P_1) - L(f, P_2) < \epsilon. \quad \square
$$