Defn. A function $f$ is said to be *differentiable at* $x_0$ if

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. In this case the limit is called the *derivative* of $f$ at $x_0$ and is denoted $f'(x_0)$.

**Note.**
1. This definition is equivalent to the requirement that the following limit exist:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

2. This, in turn, is equivalent to the following statement about how fast $f(x)$ converges to $f(x_0)$ as $x \to x_0$:

there exists a function $\eta$ such that $\lim_{x \to x_0} \eta(x) = 0$ and

$$(*) \quad f(x) - f(x_0) = (x - x_0) (f'(x_0) + \eta(x)).$$

**Examples:**
1. If $f(x) := x^2$, then $f'(x) = 2x$.
2. If $g(x) := |x|$, then $g'(0)$ does not exist.
3. If $h(x) := x|x|$, then $h'(x)$ exists and equals $2|x|$.

**Theorem.** If $f$ is differentiable at $x_0$, then $f$ is continuous at $x_0$.

**Proof.** Use $(*)$ and let $x \to x_0$. \(\square\)

**Theorem.** (Basic rules of differentiation: sums, products, quotients) Suppose that $f$ and $g$ are differentiable at $x_0$, then

1. $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.
2. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.
3. $(f/g)'(x_0) = (g(x_0)f'(x_0) - f(x_0)g'(x_0)) / g(x_0)^2$, if $g(x_0) \neq 0$.

**Theorem.** (Chain rule) If $f$ is differentiable at $x_0$ and $g$ is differentiable at $y_0 := f(x_0)$, then $h := g \circ f$ is differentiable at $x_0$ and

$$h'(x_0) = g'(f(x_0)) f'(x_0)$$
Proof. Use (*) for $f$ at $x_0$ and for $g$ at $y_0 := f(x_0)$:

\[
\frac{h(x) - h(x_0)}{x - x_0} = \frac{g(y) - g(y_0)}{x - x_0} = \frac{y - y_0}{x - x_0} (g'(y_0) + \eta_2(y)) = \frac{f(x) - f(x_0)}{x - x_0} (g'(y_0) + \eta_2(y)) = (f'(x_0) + \eta_1(x))(g'(y_0) + \eta_2(y))
\]

where $y := f(x)$. The proof is completed by using this equation, letting $x_n \to x_0$, and noticing that $y_n \to y_0$ where $y_n := f(x_n)$. \(\square\)

**Theorem.** (Rolle’s Theorem) Suppose that $\phi$ is differentiable on $(a, b)$, is continuous on $[a, b]$, and vanishes at the endpoints, then there exists $x_0$ strictly between $a$ and $b$ such that $\phi'(x_0) = 0$.

**Proof.** If $\phi$ is constant, then any point can be selected for $x_0$. Otherwise, we may assume WLOG that $\phi$ has positive values. By the Extreme Value Theorem, let $x_0$ be such that $\phi(x) \leq \phi(x_0)$ for all $a \leq x \leq b$. First, let $x_n \downarrow x_0$, then since $x_0$ gives a max, we have

\[
0 \geq \frac{\phi(x_n) - \phi(x_0)}{x_n - x_0} \to \phi'(x_0)
\]

and so, by the Squeeze Theorem, $\phi'(x_0) \leq 0$. Similarly, $\phi'(x_0) \geq 0$. \(\square\)

**Note.** Within the proof we actually established the critical point procedure of calculus: local max and min can only occur at critical points.

**Corollary.** (Mean Value Theorem) Suppose that $f$ is differentiable on $(a, b)$ and is continuous on $[a, b]$, then there exists $x_0$ strictly between $a$ and $b$ such that

\[
f'(x_0) = \frac{f(b) - f(a)}{b - a}.
\]

**Proof.** Let

\[
\phi(x) := f(x) - \left[ \frac{f(b) - f(a)}{b - a} (x - a) + f(a) \right]
\]

and apply Rolle’s theorem. \(\square\)

**Defn.** $F$ is called an antiderivative of $f$ if $F$ is differentiable and $F'(x) = f(x)$

**Corollary.** If both $F$ and $G$ are antiderivatives of $f$, then they differ by a constant, i.e. there exists a constant $c$ such that $F(x) - G(x) = c$, for all $x \in dom(f)$. 