**Defn.** A *disconnection* of a set $A$ is two nonempty sets $A_1, A_2$ whose disjoint union is $A$ and each is open relative to $A$. A set is said to be *connected* if it does not have any disconnections.

**Example.** The set $\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ is disconnected.

**Theorem.** Each interval (open, closed, half-open) $I$ is a connected set.

*Proof.* Let $A_1, A_2$ be a disconnection for $I$. Let $a_j \in A_j \neq \emptyset$, $j = 1, 2$. We may assume WLOG that $a_1 < a_2$, otherwise relabel $A_1$ and $A_2$. Consider $E_1 := \{x \in A_1 \mid x \leq a_2\}$, then $E_1$ is nonempty with $a_2$ as an upper bound. Let $a := \text{lub} E_1$. But $a_1 \leq a \leq a_2$ implies $a \in I$ since $I$ is an interval. First note that by the lemma to the least upper bound property either $a \in A_1$ or $a$ is a limit point of $A_1$. In either case, $a \in A_1$ since $A_1$ is closed relative to $I$. Since $A_1$ is also open relative to the interval $I$, then there is an $\epsilon > 0$ so that $N_\epsilon(a) \subseteq A_1$. But then $a + \epsilon/2 \in A_1$ and is less than $a_2$, which contradicts that $a$ is the lub of $E_1$. □

**Theorem.** If $A$ is a connected set, then $A$ is an interval.

*Proof.* Otherwise, there would be $a_1 < a < a_2$, with $a_j \in A$ and $a \not\in A$. But then $O_1 := (-\infty, a) \cap A$ and $O_2 := (a, \infty) \cap A$ form a disconnection of $A$. □

**Note.** Each open subset of $\mathbb{R}$ is the countable disjoint union of open intervals. This is seen by looking at open components (maximal connected sets) and recalling that each open interval contains a rational. Relatively open sets (relative with respect to $A \subseteq \mathbb{R}$) are just restrictions of these.

**Theorem.** The continuous image of a connected set is connected. The continuous image of $[a, b]$ is an interval $[c, d]$ where $c = \min_{a \leq x \leq b} f(x)$ and $d = \max_{a \leq x \leq b} f(x)$.

*Proof.* Any disconnection of the image $f([a, b])$ could be ‘drawn back’ to form a disconnection of $[a, b]$: if $\{O_1, O_2\}$ forms a disconnection for $f(I)$, then $\{f^{-1}(O_1), f^{-1}(O_2)\}$ forms a disconnection for $I = [a, b]$. So it is impossible that $f([a, b])$ is not connected. □

**Corollary.** (Intermediate Value Theorem) Suppose $f$ is a real-valued function which is continuous on an interval $I$. If $a_1, a_2 \in I$ and $y$ is a number between $f(a_1)$ and $f(a_2)$, then there exists a between $a_1$ and $a_2$ such that $f(a) = y$. 
Proof. We may assume WLOG that $I = [a_1, a_2]$. We know that $f(I)$ is a closed interval, say $I_1$. Any number $y$ between $f(a_1)$ and $f(a_2)$, belongs to $I_1 = f(I)$ and so there is an $a \in [a_1, a_2]$ such that $f(a) = y$. □

**Theorem.** Suppose that $f : [a, b] \rightarrow [a, b]$ is continuous, then $f$ has a fixed point, i.e. there is an $\alpha \in [a, b]$ such that $f(\alpha) = \alpha$.

*Proof.* Consider the function $g(x) := x - f(x)$, then $g(a) \leq 0 \leq g(b)$. $g$ is continuous on $[a, b]$, so by the Intermediate Value Theorem, there is an $\alpha \in [a, b]$ such that $g(\alpha) = 0$. This implies that $f(\alpha) = \alpha$. □