Chapter 3 deals with limits and the topology of \( \mathbb{R} \). First we recall the concept of induction.

**Theorem. (Principle of Mathematical Induction.)** Suppose that a statement \( p(n) \) is defined for each natural number \( n \). If

1. \( p(1) \) is true
2. \((p(n) \text{ true}) \implies (p(n+1) \text{ true})\) is a true statement for each \( n \in \mathbb{N} \),

then \( p(n) \) is a true statement for each natural number \( n \).

**Proof.** Suppose to the contrary that the set \( B := \{n \in \mathbb{N} \mid p(n) \text{ false}\} \) is not empty. Notice that \( 1 \not\in B \). Let \( N \) be the smallest element of \( B \) (possible since you can take a minimum of a finite set of integers), then set \( n := N - 1 \). Observe by assumption (1) that \( n \in \mathbb{N} \), and by the definition of \( B \) that \( p(n) \) is true. By assumption (2), it follows that \( p(n+1) \) is true. But \( n + 1 = N \). Contradiction. Hence \( B \) must be empty. \( \square \)

**Example.** These will useful in our study of convergence. Both are proved by induction.

1. \( \sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1 - r} \), if \( r \neq 1 \).
2. \( 1 + na \leq (1 + a)^n \), if \( a > 0 \) & \( n \in \mathbb{N} \). (Bernoulli’s inequality)

**Defn.** If \( \epsilon > 0 \) an \( \epsilon \)-neighborhood of \( a \) is defined to be the set

\[
N_\epsilon(a) := \{x \in \mathbb{R} \mid |x - a| < \epsilon\}.
\]

Notice that \( N_\epsilon(a) = (a - \epsilon, a + \epsilon) \).

**Defn.** A sequence of real numbers is defined to be a mapping from the natural numbers \( \mathbb{N} \) to the reals and is denoted by \( a_1, a_2, a_3, \ldots \) or by \( \{a_n\}_{n=1}^{\infty} \). The following definitions are used throughout the course:

1. \( \{a_n\} \) is bounded, if \( |a_n| \leq K \), for all \( n \in \mathbb{N} \).
2. \( \{a_n\} \) is convergent to \( a \), denoted by \( \lim_{n \to \infty} a_n = a \), if each \( \epsilon \)-nbhd of \( a \) contains all but a finite number of terms of the sequence. We also use the shorter notation \( a_n \to a \) when there is no ambiguity on the indices.
Example. The following are examples of sequences:

1. $1/2, 1/3, 1/4, \ldots$
2. $1, r, r^2, r^3, \ldots$
3. $1, 1 + r, 1 + r + r^2, 1 + r + r^2 + r^3, \ldots$

Lemma. $\lim_{n \to \infty} a_n = a$ if and only if

for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that if $n \geq N$, then $|a_n - a| < \epsilon$.

In short hand this reads ‘$\forall \epsilon > 0$, $\exists N = N(\epsilon) \in \mathbb{N} \ni n \geq N(\epsilon) \implies |a_n - a| < \epsilon$.’

Proof. Notice that if a statement is true except for at most a finite number of terms, then there is a largest integer for which it is not true. Take $N$ to be that integer’s successor. $\square$

Example.

1. $\lim_{n \to \infty} \frac{1}{n} = 0$.

   Proof. Use the Archimedean Principle.

2. $\lim_{n \to \infty} \frac{3n^2 - 1}{n^2 + n + 25} = 3$.

   (Hint: For a given $\epsilon > 0$, use $N := \max\{76, 4N_1\}$ where $N_1$ is the ‘cutoff’ for Example 1, i.e. any integer larger than $1/\epsilon$)

3. If $|r| < 1$, then $r^n \to 0$.

   Proof. If $r = 0$, then the conclusion follows straight away. Suppose that $0 < |r| < 1$, then if $b := 1/|r| - 1$ we see that $b > 0$ and $|r| = 1/(1 + b)$. By Bernoulli’s inequality, $|r^{n-1}| = (1 + b)^n \geq 1 + nb$. Inverting this inequality gives $|r^n - 0| \leq 1/(1 + nb)$. By example 1, pick $N$ so that $1/n < b\epsilon$ if $n \geq N$. Hence,

   $$|r^n - 0| \leq \frac{1}{1 + nb} < \frac{1}{nb} < \epsilon,$$

   if $n \geq N$. $\square$

4. $\lim_{n \to \infty} a_n = 1/(1 - r)$, if $a_n := 1 + r + r^2 + \cdots + r^n$ and $|r| < 1$.

   Proof. If $r = 0$, the conclusion follows immediately. We may suppose then that $0 < |r| < 1$. In this case, we use the identity above, i.e.

   $$a_n := \sum_{j=0}^{n} r^n = \frac{1 - r^{n+1}}{1 - r}$$
to see that
\[ a_n - a = -r^{n+1}/(1 - r) \]
where \( a := 1/(1 - r) \). Now, given \( \epsilon > 0 \), by example 3 there is an \( N_0 \) such that \( n \geq N_0 \) implies \( |r^n| < (1/|r|)\epsilon \). Combined with the displayed equation, this gives \( |a_n - a| < \epsilon \) if \( n \geq N_0 \). \( \square \)

**Theorem.** If \( \lim_{n \to \infty} a_n \) exists, then it is unique.

**Proof.** Suppose that \( \lim_{n \to \infty} a_n = A_1 \) and \( \lim_{n \to \infty} a_n = A_2 \) and that \( A_1 \neq A_2 \). Set \( \epsilon := |A_1 - A_2|/2 \). Now \( \epsilon > 0 \) so there exists \( N_1 \), such that if \( n \geq N_1 \) then \( |a_n - A_1| < \epsilon \). Since the sequence converges to \( A_2 \), we also have that there exists \( N_2 \), such that if \( n \geq N_2 \) then \( |a_n - A_2| < \epsilon \). Let \( N := N_1 + N_2 \), then \( N \) is larger than both \( N_1 \) and \( N_2 \) and so
\[ |A_1 - A_2| \leq |A_1 - a_N| + |A_2 - a_N| < 2\epsilon = |A_1 - A_2|, \]
which gives a contradiction. \( \square \)

**Theorem.** Each convergent sequence is bounded.

**Proof.** Suppose that \( \lim_{n \to \infty} a_n = a \). Let \( \epsilon := 1 \), then there is an integer \( N \) such that \( a_n \in N_\epsilon(a) \) if \( n \geq N \). This means that \( a - 1 < a_n < a + 1 \), if \( n \geq N \). If \( M := \max\{a + 1, a_1, a_2, \ldots, a_{N-1}\} \) and \( m := \min\{a - 1, a_1, a_2, \ldots, a_{N-1}\} \), then
\[ m \leq a_n \leq M, \text{ for all } n. \] \( \square \)

**Note.** Not every bounded sequence is convergent. For example, the sequence \( a_n := (-1)^n \) is bounded, but the sequence is not convergent. To see this take \( \epsilon = 1 \).