1. For a set $E$, consider the **discrete metric** given by

$$d(p, q) := \begin{cases} 
0, & \text{if } p = q \\
1, & \text{otherwise.}
\end{cases}$$

Prove that $(E, d)$ is a metric space.

2. Let

$$\phi(x) := \frac{x}{1 + x}.$$ 

Assuming that $\phi(a + b) \leq \phi(a) + \phi(b)$ and that $\phi(a) \leq \phi(b)$, when $0 \leq a \leq b$, prove that

$$d(p, q) := \phi(|p - q|)$$

is a metric for the real numbers. (Verify on your own the two assumptions on the function $\phi$.)

3. For vectors in the plane $\mathbb{R}^2$, consider the **city block metric** given by

$$d(x, y) := \sum_{j=1}^{2} |x_j - y_j|$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Show that this is in fact a metric. Can you see how to extend the definition to higher dimensions, i.e. for $\mathbb{R}^d$?

4. Consider the set $E$ of real-valued functions on $[a, b]$ and define the **uniform metric** $d$ on $E$ by

$$d(f, g) := \text{lub} \{ |f(x) - g(x)| : a \leq x \leq b \}.$$ 

Show that $d$ is in fact a metric.