COMPLEX VARIABLES
(MATH 552 – 7521)
TEST 1 – SEPTEMBER 21, 2000

Name: ____________________________

Solution

Directions: Answer all questions in the space provided. You can also use the back of the facing opposite page if you need more room.

1. Simplify the following expressions into the form of \(a + bi\) with \(a, b\) real numbers:
   a.) \(\frac{(1 - i)^2}{1 + i} = \frac{1}{2}\frac{(1 - 2i)(1 + i)}{1 + 1} = \frac{1 - i}{2}\)
   b.) \((i + 1)^{15} = (\sqrt{2} e^{i\frac{π}{4}})^{15} = 128(1 - i)\)

2. Sketch each of the following subsets of the complex numbers. For each one, determine the set's interior points and limit points. Determine if the set is open, closed, or a domain.
   a.) \(\{z \in \mathbb{C} | 1 \leq |z - 2i| < 3\}\)
      \[
      \text{interior pts.: } \{z \in \mathbb{C} | |z - 2i| < 3\}
      \]
      \[
      \text{limit pts.: } \{z \in \mathbb{C} | |z - 2i| = 3\}
      \]
      The set is neither open (some points are not interior pts.) nor closed (some of its interior points are not in the set).
   b.) \(\{z \in \mathbb{C} | \text{Re}(z) > 0\} \cup \{z \in \mathbb{C} | |z| < 1\}\) = \(S\)
      \[
      \text{interior } = S
      \]
      \[
      \text{set of limit points } = \{z \in \mathbb{C} | |z| \geq 2\} \cup \{z \in \mathbb{C} | |z| = 0\}
      \]
      i. Set is open, connected, and a domain.
3. Compute all fourth roots of $-8\sqrt{3} + 8i$. Use the polar form:

$$z = -8\sqrt{3} + 8i = r e^{i\theta} \quad r = 16, \quad \theta = \frac{5\pi}{6}$$

$$z^{\frac{1}{4}} = \sqrt[4]{16} e^{i\frac{5\pi}{6} + 2k\pi} \cdot \frac{1}{4} = \left(2 e^{\frac{5\pi}{6}}\right)^{\frac{1}{4}} w_j, \quad j = 0, 1, 2, 3$$

where $w_j$'s are the 4th roots of unity:

$$w_0 = 1, \quad w_1 = e^{\frac{2\pi}{4} i} = -1, \quad w_2 = e^{\frac{3\pi}{4} i} = i, \quad w_3 = e^{\frac{4\pi}{4} i} = -i$$

4. a.) Give the definition of the limit of a function at a point $z_0$: $\lim_{z \to z_0} f(z) = L$.

For each $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |z - z_0| < \delta$, then $|f(z) - L| < \varepsilon$.

b.) Determine $\lim_{n \to \infty} z_n$ if $z_n = \frac{n(3 + i)}{n + 1}$.

$$\lim_{n \to \infty} z_n = \lim_{n \to \infty} \left(\frac{1}{1 + \frac{i}{n}}\right)(3 + i) = 3 + i$$

5. Using the definition of the derivative, show that $f'(z) = 2z$ if $f(z) = z^2$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0$$
6. Using the Cauchy Riemann conditions, determine which of the following functions are analytic. For those functions that are analytic, compute $f'(z)$. (Note: $z = x + iy$).

(a) $(x^3 - 3xy^2) + i (3x^2y - y^3)$

$u(x,y) = x^3 - 3xy^2$, $v(x,y) = 3x^2y - y^3$ both have continuous partials in $C$.

$u_x = 3x^2 - 3y^2$, $v_y = 3x^2 - 3y^2$ so $u_x = v_y$.

$u_y = 6xy$, $u_x = -6xy$ so $v_x = -u_y$.

Cauchy-Riemann conditions hold so $f$ is analytic on $C$.

In this case, $f'(z) = u_x + iv_x = (3x^2 - 3y^2) + i(6xy)$.

(b) $e^{-x} \cos(y) + i e^x \sin(y)$

$u_x = -e^{-x} \cos(y)$, $u_y = e^x \sin(y)$ so $u_x = v_y$ only when $\cos(y) = 0$ in $y = \pi + k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$). Hence $f$ is not analytic anywhere.

7. Determine the domain of each function and at which points it is analytic.

(a) $f(z) = e^{-z} = e^{-x} \cos(y) - i e^{-x} \sin(y)$ is defined on $\Omega = C$.

By our properties for differentiation, $f'(z)$ is analytic and $g(w) = e^w$ is analytic so the composite function is analytic at each point in $\Omega$.

(b) $g(z) = \frac{2z - 3}{1 + z^2}$

Domain of this rational function is all $z$ for which the denominator does not vanish, i.e., $z \neq \pm i$.

In this case, by the quotient rule, $g$ is analytic on its domain.