Math 524
Newton's Method in One Dimension

Newton's method is an iterative method which produces a sequence of points $x_k, k = 1, 2, \ldots$ which under the right circumstances will converge to a minimizer of f. Newton's method was constructed by using the minimizer of a second order polynomial 'close to f' as a better approximation to the minimizer of f. The Taylor polynomial is a natural candidate since it 'fits' the value of f and its first two derivatives at a current approximation x_k:

$$P_{x_k}(x) = f(x_k) + f'(x_k)(x - x_k) + f''(x_k)\frac{(x - x_k)^2}{2}.$$

P_{x_k} has a critical point at

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}.$$

This is of course the Newton algorithm for finding a zero of the function $g := f'$. The following example illustrates the quadratic convergence of the method (where $e_k := x_k - x^*$):

$$|e_{k+1}| \leq C|e_k|^2.$$

Example. To demonstrate Newton's search algorithm for the function $\phi(x) = x^2 + 4cos(x)$, we have chosen a starting value of $x_0 = 3$ and $\epsilon = 10^{-10}$. Observe that the convergence is **quadratic**. The columns are respectively, the sequence of iterates, the function evaluation at the iterate , the new change in x, the value of the derivative at the iterate.

x	$\phi(x)$	$x_{\text{new}} - x_{\text{old}}$	$\phi'(x)$
2.08799541270138	2.3819e+00	-9.1200e-01	5.4355e+00
1.91222925802581	2.3173e+00	-1.7577e-01	6.9916e-01
1.89565262754691	2.3168e+00	-1.6577e-02	5.5355e-02
1.89549428154057	2.3168e+00	-1.5835e-04	5.1885e-04
1.89549426703398	2.3168e+00	-1.4507e-08	4.7525e-08
1.89549426703398	2.3168e+00	0	0

Theorem. Suppose $f^{(3)}$ is continuous and bounded by M_3, that $|f''| \geq m_2 > 0$, then for any starting value x_0, the sequence of iterates satisfies

$$|x_{k+1} - x^*| \leq \frac{M_3}{2\, m_2}\, |x_k - x^*|^2.$$

Local versions also hold after minor alterations.
Proof. We let $g := f'$, then since x^* is a critical point $g(x^*) = 0$ and so using the definition of the iteration procedure we obtain

$$\begin{aligned} x^* - x_{k+1} &= x^* - x_k + \tfrac{g(x_k) - g(x^*)}{g'(x_k)} \\ &= -\tfrac{g(x^*) - [g(x_k) + g'(x_k)(x^* - x_k)]}{g'(x_k)} \\ &= -\tfrac{g''(z)(x^* - x_k)^2}{2g'(x_k)} \end{aligned}$$

where the last equality is one of our forms for the Taylor approximation error. Using the fact that $|g''| \leq M_3$ and $|g'| \geq m_2$ gives the desired quadratic error estimate. □