Math 524
An Unconstrained Geometric Programming Example

The following example uses the dual geometric program and equality in the primal-dual inequality to solve the primal problem. We make use of the important transform $\delta_i^* = u_i(\mathbf{t}^*)/g(\mathbf{t}^*)$.

Example. Consider the problem studied earlier:

$$\begin{array}{ll} \text{Minimize:} & S(x,y,z) = xy + 2xz + 2yz \\ \text{Subject to:} & xyz = V \\ & x,y,z > 0 \end{array}$$

Using the first constraint to eliminate z, we obtain the following unconstrained geometric program:

$$\begin{array}{ll} \text{Minimize:} & g(\mathbf{t}) = t_1 t_2 + 2V/t_2 + 2V/t_1 \\ \text{Subject to:} & \mathbf{t} > \mathbf{0} \end{array}$$

which has dual program

$$\begin{array}{ll} \text{Maximize:} & v(\boldsymbol{\delta}) = \left(\frac{1}{\delta_1}\right)^{\delta_1} \left(\frac{2V}{\delta_2}\right)^{\delta_2} \left(\frac{2V}{\delta_3}\right)^{\delta_3} \\ \text{Subject to:} & \delta_1 + \delta_2 + \delta_3 = 1, \quad \boldsymbol{\delta} > \mathbf{0} \\ & \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \boldsymbol{\delta} = \mathbf{0}. \end{array}$$

The feasible vectors $\boldsymbol{\delta}$ must then satisfy the augmented matrix equation:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ \hline 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

which, applying elementary row operations, becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right].$$

There is exactly one feasible vector $\boldsymbol{\delta}^* = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$, which is then the unique maximizer for v, giving the value

$$v(\boldsymbol{\delta}^*) = 3(2V)^{\frac{2}{3}}.$$

Our theorem also states that in this case the primal solution will have a solution $\mathbf{t}^*$ and that equality holds for the primal-dual inequality, i.e.:

$$g(\mathbf{t}^*) = v(\boldsymbol{\delta}^*). \tag{1}$$

To find the minimizer $\mathbf{t}^*$, we use the nonlinear transformation equations:

$$u_i(\mathbf{t}^*) = \delta_i^*\, v(\boldsymbol{\delta}^*), \quad (i = 1, 2, n). \tag{2}$$

Recall, that this together with (1) also shows that in the optimal value of g the contribution of each of these terms to the total is then $\frac{1}{3}$.

In our case the equations (2) are

$$\begin{array}{rcl} t_1^* t_2^* & = & \frac{1}{3}\, 3(2V)^{\frac{2}{3}} \\ 2V/t_2^* & = & \frac{1}{3}\, 3(2V)^{\frac{2}{3}} \\ 2V/t_1^* & = & \frac{1}{3}\, 3(2V)^{\frac{2}{3}}. \end{array}$$

Solving (by taking logarithms in general), we obtain the minimizer for the primal problem

$$\mathbf{t}^* = (2V)^{\frac{1}{3}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Substituting back into original problem, we find that the surface area S is minimized by

$$\begin{array}{rcl} x^* & = & (2V)^{\frac{1}{3}} \\ y^* & = & (2V)^{\frac{1}{3}} \\ z^* & = & \frac{1}{2}\,(2V)^{\frac{1}{3}} \end{array}$$

with minimal value $S(x^*, y^*, z^*) = 3(2V)^{\frac{2}{3}}$. $\square$