Probability
(Math/Stat 511)
Test 1 – February 14, 2001

1	(12 *pts*)
2	(12 *pts*)
3	(16 *pts*)
4	(18 *pts*)
5	(12 *pts*)
6	(18 *pts*)
7	(12 *pts*)

Name: ______________________________ **Math/Stat?** ____

Directions: Answer all questions in the space provided. You can also use the back of the facing opposite page if you need more room. Calculators are allowed, but you must show intermediate work for partial credit.

1. Let S be an outcome (or sample) space.
   a.) Give the definition of a *probability* P on S.

   Defn: A function P from the set of subsets of a finite outcome space S to the real numbers is called a *probability* on S if it satisfies the conditions

   (a) $P(E) \geq 0$ for every subset (i.e. event) of S.

   (b) $P(S) = 1$

   (c) If events A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$.

   b.) Give the definition of a *random variable* X on an outcome space S.

   Defn: A *random variable* X on an outcome space S is any function from S into the real numbers.

2. State and prove Bayes' Theorem.

   See Course Notes or Text.

3. Consider a standard 52 card deck of playing cards. Assume that hands are dealt at random and without replacement. (In this problem do not worry about simplifying the arithmetic expressions.)

a.) What is the total number of 2 card hands that can be dealt from this deck?

Soln: $\binom{52}{2} = 26 \cdot 51 = 1,326$

b.) If 2 cards are dealt at random, what is the probability that they belong to the same suit?

Soln: $\dfrac{\binom{4}{1}\binom{13}{2}}{\binom{52}{2}}$

c.) If 5 cards are dealt at random, what is the probability that exactly three of the cards belong to the same suit?

Soln: $\dfrac{\binom{4}{1}\binom{13}{3}\binom{39}{2}}{\binom{52}{5}}$

d.) What is the probability that the five card hand is a full house?

Soln: $\dfrac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$

4. Let A, B be events in an outcome space S with a probability P.

a.) Give the definition for A and B to be independent events.

Defn: Events A and B are said to be *independent* if the condition $P(A \cap B) = P(A)P(B)$ is satisfied.

b.) If A and B are independent and $P(A) > 0$, then prove that $P(B|A) = P(B)$.

Proof: If $P(A) > 0$, then $P(B|A) := \dfrac{P(A \cap B)}{P(A)} = \dfrac{P(A)P(B)}{P(A)} = P(B)$.

c.) If A and B are independent with $P(A) = .9$ and $P(B) = .2$, then compute $P(A \cup B)$ and $P(B - A)$.

Soln: $P(B - A) := P(B \cap A') = P(B)P(A') = P(B)\,(1 - P(A))$ by the properties of independence of B and A'. So $P(B - A) = .2(.1) = .02$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = .9 + .2 - .9(.2) = .92$

5. A new test for Alzheimers' Disease is 95% effective in correctly predicting when a patient has the disease. On the other hand the rate of false positives is 10%. Assuming 4% of the population under investigation is known to have the disease, what is the probability that a patient from this population who tests positive has Alzheimers'?

Soln: Let B be the event the patient has the disease, and A the event that the test is positive for the patient, then $P(B) = .04$, $P(B') = .96$, and the priors equal $P(A|B) = .95$ and $P(A|B') = .10$. By Bayes' theorem,

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B')P(B')}$$

and so $P(B|A) = \dfrac{.95(.04)}{.95(.04) + .10(.96)} = \dfrac{.038}{.134} \approx .2836$.

6. A box of candy hearts contains 50 hearts of which 20 are white, 15 are pink, 10 are red, and 5 are purple. If 10 hearts are chosen at random from the box, what is the probability of each of the following? **(Do not take the time to simplify the arithmetic.)**

a.) there is exactly 1 red heart chosen?

Soln: $\dfrac{\binom{10}{1}\binom{40}{9}}{\binom{50}{10}}$

b.) there is no more than 1 red heart chosen?

Soln:
$$\begin{aligned} P(\#\text{ red hearts} \leq 1) &= P(\#\text{ red hearts} = 1) + P(\#\text{ red hearts} = 0) \\ &= \frac{\binom{10}{0}\binom{40}{10}}{\binom{50}{10}} + \frac{\binom{10}{1}\binom{40}{9}}{\binom{50}{10}} \end{aligned}$$

c.) there are 5 white, 2 pink, and 3 red hearts chosen?

Soln: $\dfrac{\binom{20}{5}\binom{15}{2}\binom{10}{3}}{\binom{50}{10}}$

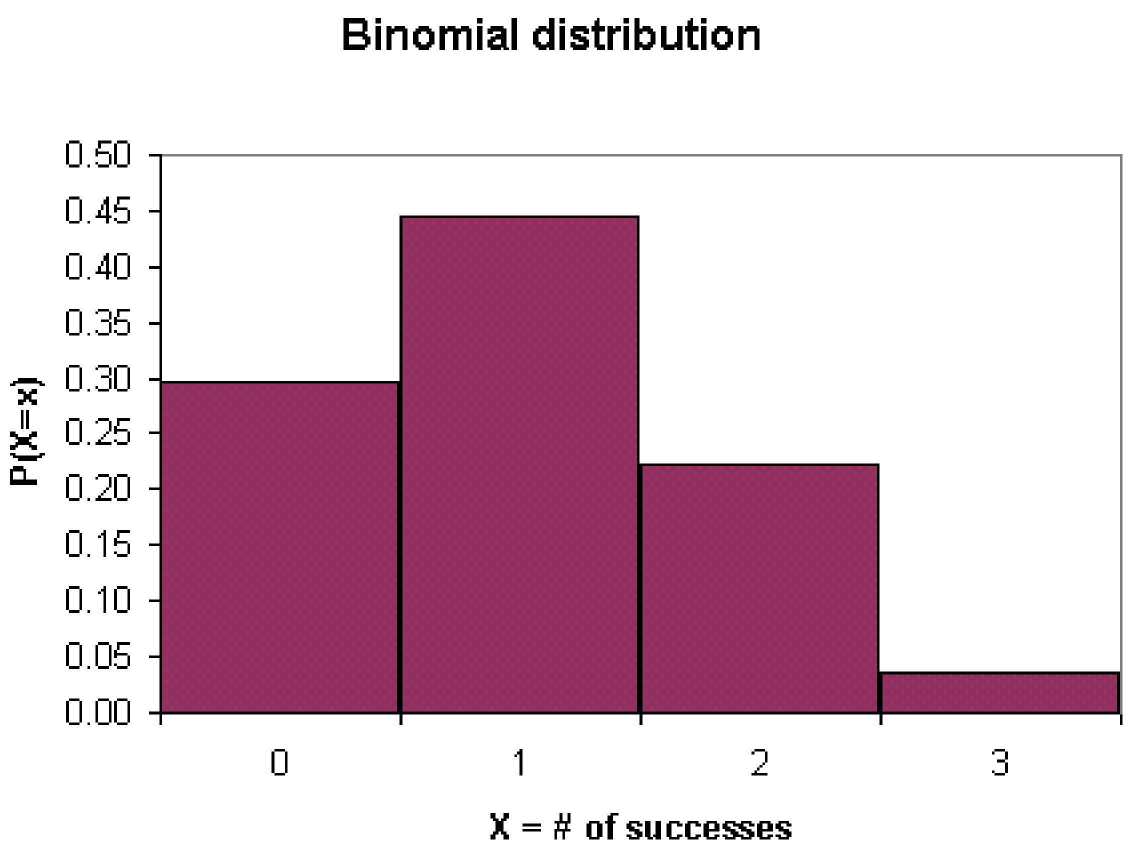
7. An experiment consists of randomly tossing a biased coin 3 times. The probability of heads on any particular toss is known to be $\frac{1}{3}$.

a.) Let X denote the number of heads, then find the probability that $X = 2$.

Soln: $P(X=2) := P(\{s \in S | X(s) = 2\}) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1$

b.) Plot the probability mass distribution function for X.

Solution:



Probability Mass Function for X

c.) Compute $P(1 \le X < 3)$.

Soln: $P(1 \le X < 3) = P(X=1) + P(X=2) = \frac{4}{9} + \frac{2}{9} = \frac{2}{3}$