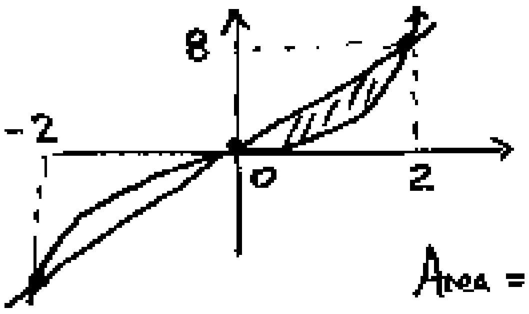
MATH 142 - CALCULUS II (SECTIONS 11-12)
FINAL EXAM – APRIL 30, 2005

Directions: Calculators will be allowed this Test. To receive proper credit however, you must show your intermediate work and *box* your final answer.

1. Compute the area bounded between the curves $y = x^3$ and $y = 4x$.



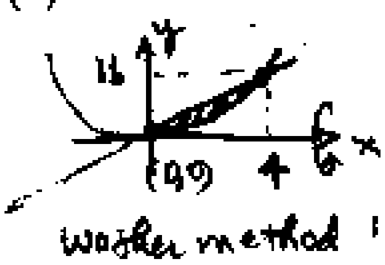
Area bounded by curves = 2 × area

Intersection: $x^3 = y = 4x \Rightarrow x^3 - 4x = 0$
$\Rightarrow x(x-2)(x+2) = 0$
$x = 0, \pm 2$

Area = $2 \cdot \int_0^2 (\underset{\text{top}}{4x} - \underset{\text{bottom}}{x^3})\,dx = 2\cdot\left(\frac{4x^2}{2} - \frac{x^4}{4}\right)\Big|_0^2 = \boxed{8 \text{ sq. units}}$

2. Compute the volume of revolution of each:

(a) about the x-axis of the region bounded by the curves $x = \sqrt{y}$ and $x = y/4$.



$\sqrt{y} = y/4$ or $y = x^2$ & $y = 4x \Rightarrow x^2 - 4x = 0 \Rightarrow x = 0, 4$

washer method: Vol = $\int_0^4 \pi (4x)^2 - \pi (x^2)^2\,dx$
$= \left(16\pi \frac{x^3}{3} - \frac{\pi x^5}{5}\right)\Big|_0^4 = \boxed{\frac{2^{11}\pi}{15}}$ cu. units

(b) about the y-axis of the region bounded by the curves $y = x^2$, $y = 0$ and $x = 1$.

cylindrical shells: Vol = $2\pi \int_0^1 x \cdot x^2\,dx$
$= \frac{2\pi}{4} x^4 \Big|_0^1 = \boxed{\frac{\pi}{2}}$ cu. units

3. Differentiate $y = \exp(x \ln(x^2 + 1))$.

Chain rule: $y' = \boxed{\exp(x \cdot \ln(x^2+1)) \cdot \left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right)}$

4. Compute each of the following integrals:

(a) $\int x^2 \arctan(2x)\, dx$

integrate by parts & partial fractions $\boxed{\frac{1}{3}x^3\arctan(2x) - \frac{1}{12}x^2 + \frac{1}{48}\ln(1+4x^2) + c}$

$\int x^2 \arctan(2x)\, dx = \int \arctan(2x)\, d\left(\frac{x^3}{3}\right) = \frac{x^3}{3}\arctan(2x) - \int \frac{x^3}{3}\,\frac{2}{1+(2x)^2}\,dx$

$= \cdots$

(b) $\int \sin(\frac{1}{2}x)\cos(2x)\, dx$

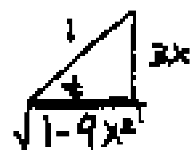
Use formula $\sin(ax)\cos(bx) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$

& integrate to get $\boxed{-\frac{1}{5}\cos(\frac{5}{2}x) + \frac{1}{3}\cos(\frac{3}{2}x) + c}$

(c) $\int \frac{1}{(1-9x^2)^{\frac{3}{2}}}\, dx$

let $9x^2 = \sin^2 t$ or $x = \frac{\sin t}{3}$, then $dx = \frac{1}{3}\cos t\, dt$

$\int \frac{\frac{1}{3}\cos t\, dt}{(1-\sin^2 t)^{3/2}} = \frac{1}{3}\int \frac{\cos t}{\cos^3 t}\, dt = \frac{1}{3}\int \sec^2 t\, dt = \frac{1}{3}\tan t + c$



$= \boxed{\frac{1}{3}\,\frac{3x}{\sqrt{1-9x^2}} + c}$

(d) $\int \frac{x}{(x^2+1)}\, dx$

$u = x^2+1$

$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + c = \boxed{\frac{1}{2}\ln(x^2+1) + c}$

(e) $\int \frac{2x^2+3}{x(x-1)^2}\, dx$

$\frac{2x^2+3}{x(x-1)^2} = \frac{3}{x} + \frac{(-1)}{x-1} + \frac{5}{(x-1)^2}$ by partial fractions

integral $= \boxed{3\ln|x| - \ln|x-1| - 5/(x-1) + c}$

(f) $\int \frac{2x^2+3}{x(x^2-1)}\, dx$

$\frac{2x^2+3}{x(x^2-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} = \frac{-3}{x} + \frac{5/2}{x+1} + \frac{5/2}{x-1}$

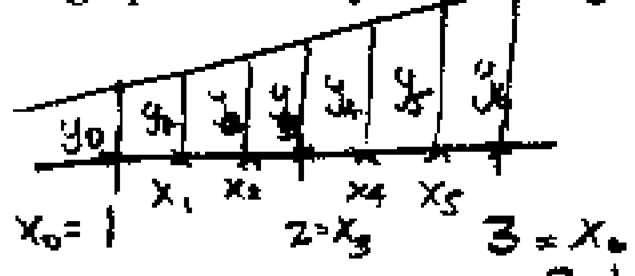
integral $= \boxed{-3\ln|x| + \frac{5}{2}\ln|x^2-1| + c}$

(g) $\int_{-1}^{1} \frac{1}{\sqrt{1-x}}\, dx$

improper integral $\lim_{b\to 1^-} \int_{-1}^{b} \frac{1}{\sqrt{1-x}}\,dx = \lim_{b\to 1^-}\left(-2\sqrt{1-x}\,\Big|_{-1}^{b}\right)$

exists and equals $\boxed{2^{3/2}}$

5. Use Simpson's rule to estimate the integral $\int_1^3 (2+x)\,dx$ with $2n = 6$ subintervals. graph of what you are doing.



$2n \to 6 \Rightarrow \Delta x = \frac{3-1}{6} = 1/3$

n	x_n	y_n	weights
0	1	3	1
1	4/3	2+4/3	4
2	5/3	2+5/3	2
3	2	4	4
4	7/3	2+7/3	2
5	8/3	2+8/3	4
6	3	5	1

$$\frac{1}{3}\,\frac{b-a}{2n}\left(y_0+4y_1+2y_2+4y_3+2y_4+4y_5+y_6\right) = \_\_\_\_$$

6. Determine if the sequence $\left\{\frac{n}{3n-1}\right\}_{n=1}^{\infty}$ converges. Justify your answer and compute if it exists.

A number of ways: eg. $f(x) = \frac{x}{3x-1} \Rightarrow f'(x) = \frac{x\cdot 3 - (3x-1)\cdot 1}{(3x-1)^2} = \frac{1}{(3x-1)^2} > 0$

$\therefore \{a_n\}_{n\geq 1}^{\infty}$ increases $0 \le a_n = \frac{n}{3n-1} < \frac{1}{3}$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{3-\frac{1}{n}}\right) = \frac{1}{3}$

7. Compute the Taylor polynomial of degree 5 of the function $f(x) = e^{2x+1}$ about $x_0 = 0$.

$$e^{2x+1} = e \cdot \exp(2x)$$

$$= e\left(1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots \frac{(2x)^5}{5!}\right) + O(x^6)$$

$$= \boxed{e + (2e)\cdot x + (2e)\cdot x^2 + \left(\frac{4e}{3}\right)x^3 + \frac{2^4 e}{4!}x^4 + \frac{2^5 e}{5!}x^5}$$

+ error

8. Determine for what values of a the series $\sum_{k=1}^{\infty}(2/a)^k$ converges.

Geometric Series converges if $|2/a| < 1 \Rightarrow \boxed{a>2 \text{ or } a<-2}$

$r = 2/a$ diverges if $|2/a| \geq 1$

9. Determine if each of the series converges and state the 'Test' you are using and criteria. No credit can be given unless the proper test is applied:

(a) $\sum_{k=1}^{\infty} \frac{1}{2k+1}$

$\boxed{\text{Diverges}}$ by comparison &/or Integral test.

e.g. $\int_1^{\infty} \frac{dx}{2x+1} = \lim_{b\to\infty}\left(\ln|2b+1| - \ln|3|\right) = +\infty$

(b) $\sum_{k=1}^{\infty}(-1)^k \underbrace{\frac{256+e^3}{\sqrt{k}}}_{C_k}$ $\boxed{\text{Converges by alternating series test}}$

$C_k = (256+e^3)\frac{1}{\sqrt{k}} \downarrow 0$ as $k \to \infty$.

(c) $\sum_{k=1}^{\infty} \frac{k}{2k+1}$ Diverges since $\lim_{k\to\infty} u_k = \lim_{k\to\infty}\frac{1}{2+1/k} = \frac{1}{2} \neq 0$

$\boxed{\text{Divergence Test}}$

10. Determine the interval of convergence for each of the following series:

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k(2x)^k}{k!}$ $= \sum_{k=1}^{\infty} \frac{(-2)^k}{k!}(x-0)^k$: $c_k = \frac{(-2)^k}{k!}$, $x_0 = 0$

$$\alpha = \lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right| = \lim_{k\to\infty}\left|\frac{\left(\frac{(-2)^{k+1}}{(k+1)!}\right)}{\left(\frac{(-2)^k}{k!}\right)}\right| = \lim_{k\to\infty}\frac{2}{k+1} = 0$$

$\Rightarrow R = \infty$

Interval of convergence $\boxed{(-\infty, \infty)}$

(b) $\sum_{k=1}^{\infty} \frac{(-1)^k}{3^k k^2}x^k$ $= \sum_{k=1}^{\infty} \frac{(-1)^k}{3^k k^2}(x-0)^k$: $c_k = \frac{(-1)^k}{3^k k^2}$, $x_0 = 0$

$$\beta = \lim_{k\to\infty}\sqrt[k]{|c_k|} = \lim_{k\to\infty}\sqrt[k]{\frac{1}{3^k k^2}} = \frac{1}{3}\cdot 1$$ (since $\lim_{k\to\infty}\sqrt[k]{k} = 1$)

$\Rightarrow R = 1$

$\boxed{(-1,1)}$