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Decomposition of functions into pairs of intrinsic mode functions

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The intrinsic mode functions (IMFs) arise as basic modes from the application of the empirical mode decomposition (EMD) to functions or signals. In this procedure, instantaneous frequencies are subsequently extracted from the IMFs by the simple application of the Hilbert transform, thereby providing a multiscale analysis of the signal's nonlinear phases. The beauty of this redundant representation method is in its simplicity and extraordinary effectiveness in many important and diverse settings. A fundamental issue in the field is to better understand these demonstrated qualities of the EMD procedures and the elementary modes they produce. For example, it is easily observed that when an EMD procedure is applied to the sum of two arbitrary IMFs, the original modes are rarely reproduced in the generated collection of IMFs. An interesting question from a representation point of view may be stated as follows: for any given sufficiently smooth function and fixed $n \geq 2$, when is it possible to represent the function as a sum of (at most) n intrinsic modes? A more interesting question is whether such a decomposition is possible when the extracted modes are constructed from a common formulation of the intrinsic properties of the function being analysed.

We provide an answer to these questions for a relaxed version of IMFs, called *weak IMFs*, which has been shown to be characterized in terms of eigenfunctions of Sturm–Liouville operators. The objective of this study is to further extend that analogy to the relationship between sums of weak IMFs and coupled Sturm–Liouville *systems*. The construction of this decomposition also provides a guide to an alternate characterization of the instantaneous frequency and bandwidth.

Keywords: intrinsic mode function; empirical mode decomposition; Sturm–Liouville systems; instantaneous frequency; redundant representations; multiresolution analysis

1. Introduction

The empirical mode decomposition (EMD) method was developed by Huang *et al.* (1998) to decompose functions into a superposition of natural modes, each of which could be easily analysed for their instantaneous frequencies and bandwidths. These natural modes, which were termed intrinsic mode functions (IMFs), were generated at each scale, going from fine to coarse, by an iterative

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procedure to locally isolate the modal behaviour. Application of EMD to real signals $f(t)$ has the purpose of representing these signals as sums of simpler modes ψ, i.e.

$$f(t) = \sum_{j=1}^{M} \psi_j(t), \tag{1.1}$$

where each ψ_j (see Cohen 1995) comes with a specific polar representation of the form

$$\psi_j(t) = r_j(t) \sin \theta_j(t). \tag{1.2}$$

These modes generalize the standard real and imaginary parts of Fourier components, in which it is required that each $r_j(t) = \hat{f}(j)$ be constant and $\theta_j(t) = jt$ be linear, but are more suitable for the study of non-stationary phase. A richer analysis of signals is provided when the amplitudes are allowed to vary and the phases are permitted nonlinear, or non-stationary, behaviour as in the representation (1.2). We wish to emphasize that even if one is provided a decomposition of a signal f in the form (1.1), for each given ψ_j the particular polar representation (1.2) that should be used is ambiguous with many possible selections of reasonable pairs of amplitudes and phases. The objective of EMD is to extract the decompositions from among such highly redundant representations so that the amplitudes $r(t)$ and the corresponding phases $\theta(t)$ at each scale are both physically and mathematically meaningful. In the case that the signal ψ is *causal*, the polar representation may be recovered by the application of an *appropriate* (cf. Sharpley & Vatchev 2006) Hilbert transform. In this case each mode ψ, after a phase shift of $\pi/2$, is represented as the real part of a complex signal Ψ

$$\Psi(t) = r(t) \exp \mathrm{i}\theta(t). \tag{1.3}$$

Obviously, the choice of amplitude–phase (r, θ) in the representation (1.3) is equivalent to the selection of an imaginary part ϕ since

$$r(t) = \sqrt{\psi^2(t) + \phi^2(t)}, \qquad \theta(t) = \arctan \frac{\phi(t)}{\psi(t)}, \tag{1.4}$$

once some care is taken to handle the branch cut.

An alternative to the Hilbert transform method for extracting instantaneous frequencies is the class of *quadrature* methods (see Cohen 1995). Whichever method is to be used should be governed by the context of the phenomena under study and the analysing procedure should produce, for each signal ψ within the signal population, a properly chosen companion ϕ for the imaginary part. This companion function ϕ should be unambiguously defined and should properly encode information about the component signal, which is in this case the IMF. Therefore, the collection of instantaneous phases present at a given instant (i.e. $t = t_0$) for a signal $f(t)$ is heavily dependent upon both the decomposition (1.1) and the selection of representations (1.2) for each mono-component. A discussion of the application of Hilbert transforms and alternative quadrature methods to IMFs to obtain instantaneous frequencies are provided in Sharpley & Vatchev (2006).

A partial sum of IMFs in the representation (1.1) can be considered as an approximation of a function f by IMFs. The stoppage criteria used in applying EMDs are numerous, but they could be considered as control of approximation errors in various metrics.

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According to Huang *et al.* (1999, p. 423), the EMD method was motivated 'from the simple assumption that any data consists of different simple intrinsic mode oscillations'. Three methods of estimating the time scales of f at which these oscillations occur have been proposed are

— the time between successive zero-crossings;
— the time between successive extrema; and
— the time between successive curvature extrema.

The use of a particular time scale is application dependent. Following the development in Huang *et al.* (1999), we define a specific class of signals with special properties that make them very well suited for phase and wave band analysis.

Definition 1.1. A function $\psi(t)$ is defined to be an *intrinsic mode function*, or more briefly an IMF, of a real variable t, if it satisfies the following two characteristic properties:

(i) ψ has exactly one zero between any two consecutive local extrema and
(ii) ψ has zero 'local mean'.

A function that is required to satisfy only condition (i) will be called a *weak IMF* and $\mathcal{W}$ will be used to designate all such functions. Note that $\psi \in \mathcal{W}$ precisely when the number of zeros and the number of extrema of ψ on I differ at most by 1.

In general, the *local mean* in condition (ii) in the EMD procedure is typically the pointwise average of the 'upper' and 'lower' envelopes as determined by the spline fits of knots consisting of, respectively, the local maxima and the local minima of ψ.

The concept of decomposing functions into oscillating modes is the essence of the Sturm–Liouville theory. In Sharpley & Vatchev (2006) we related the concept of monocomponents from phase analysis to the Sturm–Liouville theory by characterizing the weak IMFs as solutions of self-adjoint differential equations.

Theorem 1.2. *Let $\psi \in C^2(I)$ be a weak IMF with simple zeros and extrema, then there exist positive continuously differentiable functions P and Q such that ψ is the solution of the initial-value problem*

$$(Pf')' + Qf = 0, \quad f(\tau) = \psi(\tau), \quad f'(\tau) = \psi'(\tau), \tag{1.5}$$

for some $\tau \in I$.

From this result, it follows that an instantaneous frequency and bandwidth pair $(\theta', r'/r)$ can be defined implicitly through the Prüfer substitution given by

$$Pf'(t) := r(t)\cos\theta(t), \quad f(t) := r(t)\sin\theta(t). \tag{1.6}$$

The goal of this paper is to further extend the relationship between weak IMFs and eigenfunctions of self-adjoint differential operators by the representation of very general functions by superposition of weak IMFs from a common framework. Therefore, a natural question arises which must be answered if an

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EMD procedure, such as Hilbert–Huang transform (Huang *et al.* 1999), is to be realized in terms of some nonlinear minimization procedure in the context of structured dictionaries (e.g. the expository article of DeVore 1998). The question then becomes

> Under what conditions can functions be decomposed into a superposition of at most n IMFs that are inherently extracted from the function?

Here, n is a fixed number that is set in advance. We prove that any smooth function can be decomposed into two (or fewer) weak IMFs (see theorem 3.1). This is not surprising since the two IMFs are unrelated to f other than the one which should be a sinusoidal with amplitude larger than the uniform norm of f and frequency higher than any local frequency of f. In this case, there are no requirements coupling the two corresponding Sturm–Liouville operators for the ψ_j. However, with more care, weak IMF pairs ψ_1, ψ_2 can be constructed from f so that they are solutions corresponding to different eigenvalues of a Sturm–Liouville operator (SLO; with coefficients P, Q) and $f=\psi_1+\psi_2$. This is the content of theorem 3.2. The analogy between a decomposition such as EMD and coupled Sturm–Liouville systems with the same primary coefficients P and Q is provided in theorem 2.1. We prove that for any sufficiently smooth function f, there exists a coupled linear system of two differential equations, such that f represents one of the components of the solution. These coupled equations can be formulated in terms of a mechanical system of a pair of variable masses and springs and f then represents the displacement from equilibrium of the terminal mass of the system. This formulation gives an indication of a possible deeper relationship between decompositions of signals into superpositions of IMFs and concepts from system identification.

We begin by defining a general SLO that acts on sufficiently smooth functions f and is defined by

$$Lf := -\frac{1}{Q}(Pf')', \tag{1.7}$$

where P and Q are positive continuous functions on some interval $I=[a,b]$. The spectrum of L consists of all ω for which the equation $L\psi=\omega\psi$ has a solution on I that satisfies certain boundary conditions. In a more general situation the Sturm–Liouville (SL) system is defined as the solutions of the boundary-value problem (BVP)

$$\left.\begin{aligned}(P\psi')' + (\omega Q-\rho)\psi &= 0,\\ \psi(a)\cos\alpha + P(a)\psi'(a)\sin\alpha &= 0,\\ \psi(b)\cos\beta + P(b)\psi'(b)\sin\beta &= 0,\end{aligned}\right\} \tag{1.8}$$

for positive functions P and Q and real constants ω, α and β.

It is well known (see Birkhoff & Rota 1989) that (1.8) has a solution for a discrete set of eigenvalues, such that $0<\omega_1<\omega_2<\cdots$ and $\lim_{n\to\infty}\omega_n=\infty$. The corresponding eigenfunctions ψ_{ω_n} form an orthogonal basis in the weighted Hilbert space $L_2(I)(Q) := \{f : \int_I |f(t)|^2 Q(t)\,\mathrm{d}t<\infty\}$ with inner product $\int_I fgQ\,\mathrm{d}t$, and hence any function $f\in L_2(I)(Q)$ has a decomposition

$$f = \sum_{n=1}^{\infty} a_j\psi_j, \tag{1.9}$$

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with convergence in $L_2(I)(Q)$. For a fixed SL system, there exists a unique representation of any $f \in L_2$ in the form of (1.9) but, in general, requiring infinitely many terms. This should be contrasted to theorem 3.2.

2. Coupled differential systems for smooth functions

In this section, we prove the main result of this paper for representing a function as a component of a solution of a linear system of differential equations. In the proofs of the theorems, we require two technical lemmas. In order not to disturb the main flow of ideas, we refer the reader to appendix A for their proofs. Throughout this section assume f is defined on a finite interval $I=[a,b]$ and has only simple inflections and set $\Xi = \{\xi \in I : f''(\xi) = 0\}$ has $M < \infty$ elements.

Theorem 2.1. *For a function $f \in C^2(I)$ with simple extrema and for any positive numbers k_1 and k_2, there exist continuous and positive functions P and Q, depending upon f, and twice differentiable function x such that f and x satisfy the following system of differential equations*:

$$\left.\begin{aligned}(Px')' &= -k_2 Qx + k_1 Q(f-x),\\ (Pf')' &= -k_1 Q(f-x).\end{aligned}\right\} \tag{2.1}$$

Proof. For each point $\xi_j \in \Xi$, suppose $I_{j-1}=[b_{j-1},a_j]$ contains ξ_j and the initial-value problem (IVP) considered in lemma A.1 has one solution. Let $a_0 = a$, $b_M = b$ and $J_j=[a_j,b_j]$. First assume that $a_0 \in J_0$. The existence of P and Q is shown by inductively constructing them on I_0 and J_0 and repeating the same procedure on I_j, J_j for $j>0$.

For fixed k_1, k_2 positive, set

$$\lambda_1 := -\left(\frac{1}{k_1} + \frac{2}{k_2}\right) \tag{2.2}$$

and

$$\lambda_2 := \frac{1}{k_1 k_2}. \tag{2.3}$$

Since f, f' and f'' are bounded on I, then for any given real number u, it follows that $C_1 = |(b-a)u| + (|\lambda_1|/|\lambda_2|)(|f'(a)(b-a)| + |f(a)| + \|f\|_\infty)$ and $C_2 = ((b-a)^2)/|\lambda_2|\|ff''\|_\infty$ are finite constants. Pick $\tilde{T} > \max\{1, C_1 + \sqrt{C_1^2 + 4C_2}\}$. (We note that these are the primary constants and parameters in lemma A.2 of the appendix. Although somewhat redundant, introducing these here aids in motivating the proof of alternating the successive application of the two lemmas.)

On the interval J_0 the function f does not have an inflection, and hence $|f''| > \eta > 0$. By applying lemma A.2 with $T = -\mathrm{sign}(f''(a_0))\tilde{T}$ and $\phi'(a_0) = u$, we obtain a twice differentiable solution Φ that does not change sign on J_0. Furthermore, since f'' does not change its sign on J_0 and owing to the choice of the initial condition for $\Phi(a_0)$, it follows that the function $Q_0 := -(P_0 f''/\Phi)$ is positive and continuous on J_0 for any fixed function $P_0 > 0$. Fix P_0 a positive constant and define $x = f - (1/k_1)\Phi$, then it follows that

$$P_0 f'' = -k_1 Q_0 (f-x). \tag{2.4}$$

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Twice differentiating the integral equation in lemma A.2, we get $\Phi'' = -(\lambda_1/\lambda_2)f'' - (1/\lambda_2)(ff''/\Phi)$ and after substituting in $x'' = f'' - (1/k_1)\Phi''$, we obtain the following differential equation for x:

$$P_0 x'' = P_0 f''\left(1 + \frac{\lambda_1}{\lambda_2 k_1}\right) + \frac{P_0}{k_1\lambda_2} f\left(-\frac{Q_0}{P_0}\right) = Q_0(-k_2 x + k_1(f-x)). \tag{2.5}$$

By combining (2.4) and (2.5), we have constructed the desired system (2.1) for J_0.

Next we construct the functions P and Q on the interval I_0, but here the function f has an inflection point and lemma A.2 cannot be applied. Instead, on this interval this will be the role of lemma A.1. Note that the system (2.1) is equivalent to the system

$$\left.\begin{aligned} x'' + px' &= -k_2 qx + k_1 q(f-x), \\ f'' + pf' &= -k_1 q(f-x), \end{aligned}\right\} \tag{2.6}$$

where $p := (P'/P)$ and $q := (Q/P)$. Let q be the linear function connecting the points $(b_0, (Q_0(b_0)/P_0))$ and $(a_1, (|f''(a_1)|/\tilde{T}))$. Since $f' \geq \eta > 0$ on I_0, then the second equation can be solved for p

$$p = -\frac{f''}{f'} - k_1 q\left(\frac{f}{f'} - \frac{x}{f'}\right), \tag{2.7}$$

and upon substituting in the first equation of (2.6), we obtain the differential equation in lemma A.2 for x. The initial conditions naturally come from the already determined values $y_0 = x(b_0), y_1 = x'(b_0)$. On J_0 we have that $\Phi = -(1/q)f''$ and $|\Phi| \geq (\tilde{T}/2)$ including the right endpoint b_0, and hence the estimate $|q(b_0)| = (|f''(b_0)|/|\Phi(b_0)|) \leq (2|f''(b_0)|/\tilde{T}) \leq 2\|f''\|_\infty$ holds true. From the definition of q it follows that $|q(a_1)| \leq (2\|f''\|_\infty/\tilde{T})$ and since q is linear on I_0 we have that the last estimate holds on over all of I_0. All the conditions in lemma A.1 are met, and hence there exists a function x that is a solution to the differential equation (A 2). Substituting x in (2.7), we get a continuous function p on $J_0 \cup I_0$. The functions P_1 and Q_1 are determined from the system

$$\frac{P_1'}{P_1} = p,$$

$$\frac{Q_1}{P_1} = q,$$

and have solutions $P_1(t) = P_0 \exp(\int_{b_0}^t p(v)\,\mathrm{d}v)$, $Q_1(t) = P_0 q(t) \exp(\int_{b_0}^t p(v)\,\mathrm{d}v)$, where $P_0 = P_1(b_0)$ and $Q_0(b_0) = Q_1(b_0)$. We have now constructed the continuous functions P and Q on the interval $J_0 \cup I_0$. Inductively, we can continue the construction on J_1 with the same constants λ_1 and λ_2 as on J_0 and initial conditions $\phi(a_1) = -\mathrm{sign}\,(f''(a_1))(1/q(a_1)), \phi'(a_1) = u$, u arbitrary. Note that $|\phi(a_1)| = |T|$ and hence lemma A.1 provides continuous positive P_2 and Q_2 on J_1. Repeating the same procedure on I_j and J_j, $j \geq 1$, we construct continuous functions P, Q and x, which satisfy the system (2.1). This completes the proof if the partition of $I = [a,b]$ begins with J_0.

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In the case $a_0 \in I_0$, we then start by instead using lemma A.1 with appropriate choice for the initial conditions and the function q. ■

In the previous theorem, we showed the existence of continuous positive coefficients P and Q assuming that the function f has a continuous second derivative. If f has fourth-order continuous derivatives, i.e. $f \in C^4(I)$, the next result shows that we can modify our construction to guarantee in addition that the coefficient P is continuously differentiable, so that the equations (2.1) hold in the classical sense.

Theorem 2.2. *Let f be a real-valued function in $C^4(I)$. If f has only simple extrema, then for any two positive numbers k_1 and k_2 there exist positive functions P and Q such that P, P' and Q are continuous on I, and f is a solution to* (2.1).

Proof. Following the proof of theorem 2.1, it is clear that the only points where P' could have breaks are the points a_j, $j=1, 2, \ldots, M$. We show that around these points the construction of P can be adjusted in a way that provides continuous P'.

Let k_1 and k_2 be fixed and λ_1, λ_2 and $\tilde{T}$ be as in theorem 2.1, and

$$Lf = -\frac{1}{Q}(Pf')' = -mf'' - sf', \tag{2.8}$$

where $m=(P/Q)$ and $s=(P'/Q)$. The system (2.1) is equivalent to the fourth-order equation

$$f + \lambda_1 Lf + \lambda_2 L^2 f = 0, \tag{2.9}$$

where $L^2 f = L(Lf)$. After substituting (2.8) into (2.9) and isolating s'' in the resulting equation we obtain, for the given f, the second-order linear equation for s

$$s'' = F(t, s, s'), \tag{2.10}$$

where

$$\begin{aligned} F(t,s,s') = &-\frac{1}{mf'}\left(\frac{f}{\lambda_2} - \frac{\lambda_1}{\lambda_2} mf'' + m^2 f^{(4)} + 2mm' f^{(3)} + mm'' f''\right) \\ &-\frac{1}{mf'}\left(\left(-\frac{\lambda_1}{\lambda_2} f' + 2mf^{(3)} + m' f''\right)s + (sf' + 2mf'')s' + f'' s^2\right). \end{aligned}$$

In the proof of theorem 2.1, the functions P and Q were constructed continuous and bounded on I, and hence the corresponding p and q are continuous and bounded on I. Clearly $P'=(s/m)P$, and hence P, P', and Q are continuous if and only if s and m are continuous. On the other hand, $q=(1/m)$ and $p=(s/m)$. By using the existence theorem for a solution of BVP (appendix lemma A.1), we alter p and q in one-sided neighbourhoods of the points a_j in a way that p and q are continuous on the entire interval I.

For a fixed a_j, there exists a neighbourhood in which the function f does not have extremal points and inflections, and hence the function (f''/f') has a constant sign on that neighbourhood, say $\epsilon = \operatorname{sign}_{t<a_j}(f''(t)/f'(t))$. We modify q, obtained from theorem 2.1, on the interval I_j preserving all necessary conditions required for the proof of theorem 2.1. Let $e=(\tilde{T}/|f''(a_j)|)$ and $\rho=(e/4R)$, where $R>0$, then on the interval $I_{j,\rho}=[a_j-\rho, a_j]$ the straight line $m(x)=-2\epsilon R(x-a_j)+e$ is strictly positive and $(1/m) \leq (2\max_{t\in I}|f''|/\tilde{T})$. If necessary, R might be

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increased in order that $I_{j,\rho}$ be contained entirely in I_j. By using the previously constructed P and Q from theorem 2.1, the new q is modified only on the interval I_j. Namely, on $I_{j-1}\backslash I_{j,\rho}$ we define q as the linear function that connects the points $(b_j, (Q_j(b_j)/P_j))$ and $(a_j-\rho, 1/(m(a_j-\rho)))$ and on $I_{j,\rho}$ we define $q(t)=1/m(t)$. Similar to the construction in theorem 2.1, we construct the function p on $I_{j-1}\backslash I_{j,\rho}$ and let $s_0 = s(a_j-\rho) = p(a_j-\rho)m(a_j-\rho)$. By using the already defined p on J_j, let $s_\rho = s(a_j) = p(a_j)m(a_j) = p(a_j)e$. We show that on $I_{j,\rho}$, the BVP (2.10) with boundary conditions $s(a_j-\rho)=s_0, s(a_j)=s_\rho$ has a solution $s(t)=p(t)m(t)$, and hence the newly defined q and p are continuous on $I_{j-1}\cup J_j$.

Without loss of generality, we can consider $I_\rho = I_{j,\rho} = [0,\rho]$. Let $R>0$ be large enough such that $|s_0|, |s_\rho| \le R$. Since $F(t,s,s')$ is linear with respect to s' on $E(\rho, R)$ (see appendix A 1), then for a fixed R there exists constants γ and $C>0$ depending only on f and R such that $\gamma R<1$ and $|F| \le \gamma(s')^2 + C$ on $E(\rho, R)$. Next we show that the conditions for $F(t,\pm R,0)$ hold in limit for $R\to\infty$. From the choice of ρ, it follows that $|m| \le (3/2)e$ independent of R on I_ρ. Let μ be a constant with a value either 1 or -1. Since f and its derivatives do not depend on R and $m''=0$ on I_ρ, it follows that

$$\operatorname{sign} \lim_{R\to\infty} \frac{F(t,\mu R,0)}{R^2} = \operatorname{sign} \lim_{R\to\infty} \frac{f''}{mf'} \frac{2\epsilon R\mu R - R^2}{R^2} = \operatorname{sign} \epsilon(2\epsilon\mu - 1)$$
$$= \operatorname{sign} (2\mu - \epsilon) = \mu,$$

holds uniformly on I_ρ. Hence we can pick large enough R for which $F(t,R,0)\ge 0$ and $F(t,-R,0)\le 0$. With that choice of R, all the necessary conditions in lemma A.1 are met and the solution to (2.10) with $s(a_j-\rho)=s_0, s(a_j)=s_\rho$ provides a solution p.

By repeating the same considerations for all of the a_j's, we can construct positive coefficients P and Q for the system (2.1) such that P, P' and Q are continuous functions on the entire I.

Remark 2.3. In the previous discussion, we proved that for an appropriate function f there exists companion function x such that the pair (f, x) is a solution of a system of type (2.1). The function x was constructed first on neighbourhoods around the inflections of f and then, starting from the beginning of the interval I, the pieces were smoothly connected. In order to construct a particular function x, we need to specify the initial values $x(a)$ and $x'(a)$. On the other hand, we can start from any interior point $\tilde{t}\in I$ with initial values $x(\tilde{t})$ and $x'(\tilde{t})$ and construct the function on the left and on the right from $\tilde{t}$.

Furthermore, for functions that satisfy the conditions in theorem 2.2, we considered a method to smoothly connect two disjoint segments of x. Summing up, we can construct the function x to satisfy boundary–initial conditions (BVP) by initializing the 'left' branch of x at a and the 'right' branch of x at b, followed by connecting smoothly the two branches on an appropriate subinterval of I.

In the next section, the relation between the EMD and the representation of f as a solution of system of differential equations is discussed.

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3. Decomposition into pairs of weak IMFs

In the current section, we prove that any function $f\in C^2(I)$ with simple zeros and extrema can be decomposed into a sum of two or fewer weak IMFs. The decomposition is constructive and it is not obtained by applying EMD. Two types of decompositions are considered. The first one is a more general result about the existence of only two weak IMFs. In general, the two weak IMFs are not strongly related to physical properties of the function. The second method for decomposition can be associated with the displacement from equilibrium of one of the two masses in a simple mechanical system with time-dependent masses and forces. Details are given later in this section. In either case, the decompositions are non-unique. We denote the interval by $I=[a,b]$.

Theorem 3.1. *Let $f\in C^1(I)$, then there exist two weak IMFs ψ_1 and ψ_2, such that $f=\psi_1+\psi_2$ on I.*

Proof. Let ψ_1 be any weak IMF such that ψ_1' is also a weak IMF and the extrema of ψ_1 do not coincide with the extrema of f. For example, $\psi_1(t)$ could be $\sin(\mu t+\zeta)$ for appropriate choices of μ and ζ. Define $g_\alpha=f-\alpha\psi_1$ for $\alpha\in\boldsymbol{R}$. Since f is bounded on I, then for any point $t\in I$ we have that $\lim_{\alpha\to\infty}\operatorname{sign}(g_\alpha(t))=\operatorname{sign}(\psi_1(t))$ and $\lim_{\alpha\to\infty}\operatorname{sign}(g_\alpha'(t))=\operatorname{sign}(\psi_1'(t))$. Since f, f', ψ_1 and ψ_1' are continuous on I, there exists large enough α_1 for which $\psi_2=g_{\alpha_1}$ is a weak IMF and $f=\alpha_1\psi_1+\psi_2$ on I. Since $\alpha_1\psi_1$ is a weak IMF the proof is complete. ∎

From the proof, it is clear that ψ_1 and ψ_2 may not be related in general to the local properties of the function and further analysis based on that representation will not reveal any useful local characteristics of f. Another type of decomposition into weak IMFs, which does admit physical interpretation for a sufficiently smooth function f, can be obtained from theorem 2.1. That is the content of the next result.

Theorem 3.2. *Let $f\in C^2(I)$, then there exist weak IMFs ψ_1 and ψ_2, such that $f=\alpha_1\psi_1+\alpha_2\psi_2$ on I, and ψ_1 and ψ_2 are solutions of the self-adjoint differential equations $(P\psi_j')'=-\omega_jQ\psi_j$, for some positive continuous functions P and Q and positive $\omega_j, j=1,2$.*

Proof. Fix k_1, $k_2>0$. By applying theorem 2.1 to f, it follows that there exist functions P, Q and x such that the vector

$$\boldsymbol{X}=\begin{pmatrix}x\\ f\end{pmatrix},$$

is a solution of the system of linear differential equations $(PEX')'=QKX$, where

$$K=\begin{pmatrix}-(k_1+k_2) & k_1\\ k_1 & -k_1\end{pmatrix}$$

and

$$E=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

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Direct calculations show that the numbers $-\omega_1$ and $-\omega_2$, where $\omega_1 = (2k_1 + k_2 - \sqrt{4k_1^2 + k_2^2})/2 > 0$ and $\omega_2 = (2k_1 + k_2 + \sqrt{4k_1^2 + k_2^2})/2 > 0$ are the two eigenvalues of the matrix K, and hence there exists an invertible matrix of real numbers A such that $A^{-1}KA = -\Omega$, where

$$\Omega = \begin{pmatrix} \omega_2 & 0 \\ 0 & \omega_1 \end{pmatrix}.$$

Define the vector

$$\boldsymbol{\Psi} = \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} = A^{-1}X.$$

The functions ψ_1 and ψ_2 are two weak IMFs such that $f = \alpha_1\psi_1 + \alpha_2\psi_2$ for some real α's. Indeed, we have

$$\begin{aligned}(PE\boldsymbol{\Psi}')' &= (P(A^{-1}X)')' = A^{-1}(PEX')' = A^{-1}QKX \\ &= QA^{-1}KAA^{-1}X = -Q\Omega\boldsymbol{\Psi}. \end{aligned} \tag{3.1}$$

From (3.1) it follows that the functions ψ_1 and ψ_2 are solutions to the differential equations $(P\psi')' = -\omega Q\psi$ for $\omega = \omega_1$ and $\omega = \omega_2$, respectively. Hence they are weak IMFs generated by one and the same self-adjoint operator. Adding the initial conditions from f, and $x(a) = x_a, x'(a) = x'_a$, where x_a, x'_a are determined from the proof of theorem 2.1, we can determine constants α_1 and α_2 such that $f = \alpha_1\psi_1 + \alpha_2\psi_2$. ■

In general, the functions ψ_1 and ψ_2 do not satisfy the same boundary conditions, and hence they are not part of one and the same Sturm–Liouville system. In the case $f \in C^4(I)$, we can use remark 2.3 to construct ψ_1 and ψ_2 as two eigenfunctions of one and the same Sturm–Liouville system.

Corollary 3.1. *Let $f \in C^4(I)$ and have simple zeros, extrema and inflections. If ω_1 and ω_2 are as in theorem* 3.2*, then there exists a Sturm–Liouville system of type* (1.8)*, such that f is a linear combination of two, or fewer, of its eigenfunctions.*

Proof. We modify P and Q in such a way that the functions ψ_1 and ψ_2 constructed in theorem 3.2 satisfy the same boundary condition of type (1.8). From the proof of theorem 3.2, it follows that

$$\begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} = A^{-1}\begin{pmatrix} x \\ f \end{pmatrix},$$

where A^{-1} is a non-singular matrix, and hence ψ_1 and ψ_2 satisfy the same boundary conditions if and only if f and x satisfy the same boundary conditions. Since the function f is given, the parameters in the boundary condition

$$f(a)\cos\alpha + P(a)f'(a)\sin\alpha = 0, \tag{3.2}$$

$$f(b)\cos\beta + P(b)f'(b)\sin\beta = 0, \tag{3.3}$$

should be chosen appropriately. The function P is coupled with the function x but α and β could be arbitrary.

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Let x_l denote a function x that is constructed in theorem 2.2 by starting from the end point a. We show that x_l can be constructed to satisfy (3.2). Depending on the values of f and f' at a, we consider the following three possible cases.

— *Case* (*i*). The function f does not have both a zero and an extrema at a. Then, (3.2) is equivalent to $f(a)-f(a)/f'(a)f'(a)=0$, and so x must satisfy $x_l(a)-f'(a)/f(a)x_l'(a)=0$.

In the case $a\in J_0$, from theorem 2.1, it follows that x_l constructed as $x_l(t)=f(t)-(1/k_1)\Phi(t)$ satisfies $x_l(a)-(f(a)/f'(a))x_l'(a)=0$, if Φ is the solution of the initial-value integral equation (A 11) with initial condition $\Phi(a)=T$ and $\Phi'(a)=(f'(a)/f(a))T$.

In the case $a\in I_0$, from theorem 2.1, it follows that the function x_l, constructed as the solution of the IVP (A 2) with initial conditions $x_l(a)=f(a)$ and $x_l'=f'(a)$, satisfies (3.2).

— *Case* (*ii*). The function f does not have zero at a, but $f'(a)=0$. By choosing $\alpha=\pi/2$ in (3.2), the initial condition for f becomes $f'(a)=0$ and the same considerations as in case (i) lead to a construction of x_l with $x_l'=0$ and any choice of $x_l(a)$.

— *Case* (*iii*). The function f does not have extrema at a, but $f(a)=0$. In that case, we construct P with zero at a and positive elsewhere. Choosing $\alpha=\pi/2$ in (3.2), we have a singular Sturm–Liouville system. Since f has simple zeros and extrema, it follows that $f'(a)\neq 0$ and $a\in I_0$. Furthermore, by repeating the same construction of p and q as in theorem 2.2 with boundary values $s(a)=0$ and $s(\rho)=v$, for an appropriate ρ and v, we obtain a continuously differentiable non-negative function P with its only zero value at the point a. The corresponding Q is a continuous positive function.

In a similar way, starting from b, a function x_r can be constructed that satisfies the boundary conditions (3.3). Finally, x_l and x_r can be connected smoothly on an appropriate interval as in theorem 2.2 to construct a function x that satisfies (3.1) and (3.2). The resulting weak IMFs ψ_1 and ψ_2 are eigenfunctions of a Sturm–Liouville system of type (1.8) with corresponding eigenvalues ω_1 and ω_2. One immediate consequence is that ψ_1 and ψ_2 are orthogonal on I with a weight function Q. ■

The main purpose of the EMD method is to decompose a function into a sequence of IMFs in order to extract important physical characteristics from each component by the application of the Hilbert transform. Next we discuss possible physical interpretations of the decompositions considered in theorems 3.1 and 3.2.

Let $SL(P, Q)$ denote the set of all the eigenfunctions of Sturm–Liouville operators defined in (1.7). It is clear that $SL(P, Q)$ contains the eigenfunctions of any Sturm–Liouville system. For any fixed boundary conditions, the system of corresponding eigenfunctions is complete in $L_2(I, Q)$ and it follows that the set of functions $SL(P, Q)$ is complete and redundant in $L_2(I, Q)$. Furthermore, since the weight function Q is strictly positive, continuous on the finite interval I, then the space $L_2(I, Q)$ is norm equivalent to the space $L_2(I)$. Varying P and Q over all continuous positive functions on I, we observe from theorem 1.2 that $\mathcal{W}=\{\psi : \psi \in SL(P, Q), P, Q \text{ positive continuous functions on } I\}$.

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Self-adjoint differential equations describe many physical processes and have embedded physical characteristics. A typical example is the periodic BVP

$$\psi'' + \omega^2\psi = 0,$$

$$\psi(0) = \psi(2\pi), \psi'(0) = \psi'(2\pi),$$

that defines the trigonometric system $\{\cos nt, \sin nt\}_{n=0}^{\infty}$ on the interval $[0,2\pi)$. This differential equation models the frictionless displacement from equilibrium of a unit mass attached to a spring with a spring constant ω^2. If the 'periodic' boundary condition is replaced with an initial condition, then the SL operator $Lf = -f''$ generates trigonometric functions $\cos \omega t$, $\sin \omega t$ that still obey the physical relation with mass–spring system, but is a redundant system. One method to decompose a function into a linear combination of elements from a redundant system is by using 'greedy'-type algorithms (for reference see DeVore 1998 and Temlyakov 1999). Generally, the resulting representation consists of infinitely many terms.

In analogy with the trigonometric system, the characteristic equation for a weak IMF $(P\psi')' = -Q\psi$ can be interpreted as the physical model according to which a variable mass $P(t)$ is attached to a spring with a variable spring constant $Q(t)$ and vibrates frictionlessly around its equilibrium position $\psi(0)$. The solution $\psi(t)$ is the displacement of the mass at the instant t. An equivalent form of the characteristic differential equation is

$$\psi'' + p\psi' + q\psi = 0, \tag{3.4}$$

where $q > 0$ and p is an arbitrary function. In that form the mass–spring interpretation could be that $\psi(t)$ is the displacement of a unit mass attached to a spring with a variable spring constant q if the motion is subjected to a frictional force $-p\psi'$. In both interpretations, $\sqrt{Q}$ or $\sqrt{q}$ is considered the frequency of the motion. In the second case, p can be considered as the instantaneous bandwidth of ψ. The instantaneous quantities of ψ can be defined by using the Prüfer substitution (1.6).

Summarizing the results of this section, theorems 3.1 and 3.2 not only show the existence of the decomposition of a function f into finitely many weak IMFs, but they also provide an additional analytical resource to the Hilbert transform method in order to define instantaneous frequency and bandwidth by using the physical interpretation from a model system of differential equations.

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Appendix A. Technical lemmas on differential and integral equations

In this appendix, we gather several results about the existence of solutions and continuity with respect to the parameters of certain types of integral and differential equations. We have isolated these to an appendix in order not to distract from the main ideas of the decomposition theorems of §2. These results are included in order to verify that the stated dependence of the solutions upon the adjustable parameters, which are used in the constructions of the Sturm–Liouville operators, is possible.

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Throughout this section, the function f is assumed to be twice continuously differentiable on an interval $J_0=[c,d]$ with a finite number of inflections and with the property that all its zeros and extrema are simple. We need controlled solutions to two types of specialized problems that are dependent upon properties of f, in order to complete the constructions of §2. The first type provides a neighbourhood about the simple inflection points of f where a controlled solution to a differential equation of a certain form exists. The second type of problem is an integral equation that is used for subintervals that are free of inflection points of f. The constructions in §2 require these two lemmas to be used in tandem to produce second-order systems with the desired properties.

(*a*) *Differential system*

We denote the finite collection of inflection points of f by

$$\Xi = \{\xi : f''(\xi) = 0, \xi \in I\}.$$

In the next results, it is assumed that f, p, q, k_1 and k_2 are given, and in order to simplify the notation we set

$$G(t,x,z) = k_1 qf - (k_1+k_2)qx + \left(k_1 q\frac{f}{f'} + \frac{f''}{f'}\right)z - \frac{k_1 q}{f'}xz. \tag{A 1}$$

In this case, we have the following result for the solution of the differential equation with coefficients depending upon f as in expression (A 1).

Lemma A.1. *Suppose k_1 and k_2 are positive real numbers and that q is continuous on J_0 with $\|q\|_\infty < 2\|f''\|_\infty$. Suppose further that ξ is an inflection point of the twice continuously differentiable function f, such that $f'(\xi)\neq 0$, then there exists a subinterval $J=[a,b]\subset J_0$ containing ξ in its interior, such that the IVP*

$$x'' = G(t, x(t), x'(t)), \tag{A 2}$$

$$x(a) = y_0, \quad x'(a) = y_1, \tag{A 3}$$

has a solution on J for any real y_0, y_1.

Proof. Without loss of generality, we may assume from the hypothesis that f' does not vanish on J_0 and by compactness that $|f'(t)|\geq\sigma>0$, for all $t\in J_0$. Denote, respectively, the constants

$$m_0 := 2k_1\|f''\|_\infty\|f\|_\infty, \tag{A 4}$$

$$m_1 := 2(k_1+k_2)\|f''\|_\infty, \tag{A 5}$$

$$m_2 := \|2k_1|f| + 1\|_\infty \frac{\|f''\|_\infty}{\sigma}, \tag{A 6}$$

$$m_3 := 2k_1\frac{\|f''\|_\infty}{\sigma}. \tag{A 7}$$

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Further, select positive c_m such that

$$c_m^3 > \max\left(1,\ \left|m_0 - 3\frac{m_1 m_2}{m_3}\right|,\ m_3 c_m^2 + (m_1+m_2)c_m + m_1\right). \tag{A 8}$$

J is then set to the interval centred at ξ with length $\delta := (1/c_m^2)$. If necessary, c_m can be increased to ensure $\delta < 1$ and $J \subset J_0$. We show that J is the interval whose existence is stated in the lemma.

Write the second-order equation (A 2) as its equivalent first-order system in integral form,

$$\left.\begin{aligned} x(t) &= y_0 + \int_a^t z(\tau)\,\mathrm{d}\tau, \\ z(t) &= y_1 + \int_a^t G(\tau, x(\tau), z(\tau))\,\mathrm{d}\tau, \end{aligned}\right\} \tag{A 9}$$

where $G(t,x,z)$ is as in expression (A 1). A standard technique to show the existence of solutions of differential equations is the method of successive approximations. In particular, we may, without loss of generality, assume that $y_0 = y_1 = 0$ and let $x_0 = 0$, $z_0 = 0$, then inductively construct the sequence pairs

$$\left.\begin{aligned} x_{n+1}(t) &= y_0 + \int_a^t z_n(\tau)\,\mathrm{d}\tau, \\ z_{n+1}(t) &= y_1 + \int_a^t G(\tau, x_n(\tau), z_n(\tau))\,\mathrm{d}\tau. \end{aligned}\right\} \tag{A 10}$$

If we show $|x_n| \le c_m$ and $|z_n| \le c_m$ for $n = 0,1,2,\ldots$, we can use a standard argument to then establish that the limit functions x and z exist and the convergence is uniform. Using the inductive assumption $|x_n| \le c_m$ and $|z_n| \le c_m$, we show that $|x_{n+1}| \le c_m$ and $|z_{n+1}| \le c_m$. For x_{n+1}, we have $|x_{n+1}| \le |z_n|\delta \le |z_n| \le c_m$, since $\delta < 1$. For z_{n+1}, using the bound $|q| \le 2\|f''\|_\infty$, we have

$$|z_{n+1}| \le \delta(m_0 + m_1|x_n| + m_2|z_n| + m_3|x_n||z_n|).$$

The polynomial function $b(x,z) = m_0 + m_1 x + m_2 z + m_3 xz$ has only one local extremum and the value at that extremum is $m = |m_0 - 3(m_1 m_2/m_3)| < c_m^3$, from the choice of c_m. The global maxima of $b(x,z)$ on the square $|x| \le c_m, |z| \le c_m$ are either the value at the extremum or a value from the boundary. It is easy to see that the maximum value on the boundary is $b(c_m, c_m) = m_0 + (m_1+m_2)c_m + m_3 c_m^2 < c_m^3$, from the choice of c_m. Hence $|z_{n+1}| \le \delta \max|b(|x|,|z|)| \le (1/c_m^2)c_m^3 = c_m$ and the limit functions $x = \lim_{n\to\infty} x_n$, and $z = \lim_{n\to\infty} z_n$ is the solution pair of (A 9). Hence x is a solution of the IVP (A 2) and (A 3) on the interval $[a, a+\delta]$ that contains the point ξ. ■

(*b*) *Integral equation*

One additional technical lemma is required for the proof of theorem 2.1. The result establishes the existence of positive solutions of an integral equation on intervals that avoid inflections of f and will be used on subintervals complementary to those handled by lemma A.1.

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Lemma A.2. *Let $f \in C^2(I)$ with simple zeros and extrema. Let $\tilde{t} \in I$. Then for any non-zero λ_1 and λ_2 and any u, there exists a positive real number T_0, depending only on I and f, such that for each T, $|T| \geq T_0$, the integral equation,*

$$\phi(t) = T + (t-\tilde{t})u + \frac{\lambda_1}{\lambda_2}(f(\tilde{t}) + f'(\tilde{t})(t-\tilde{t}) - f(t)) - \frac{1}{\lambda_2}\int_{\tilde{t}}^{t}\int_{\tilde{t}}^{\tau}\frac{f(v)f''(v)}{\phi(v)}\,\mathrm{d}v\,\mathrm{d}\tau, \tag{A 11}$$

has a solution Φ, such that $|\Phi| > (|T|/2)$ on I, with $\phi(\tilde{t}) = T$, $\phi'(\tilde{t}) = u$.

Proof. We prove the lemma in the case $\tilde{t} = a$, where $I = [a,b]$. The case $\tilde{t} \in (a,b]$ is similar. Let $c_1 = |(b-a)u| + (|\lambda_1|/|\lambda_2|)(|f'(a)(b-a)| + |f(a)| + \|f\|_\infty)$ and $c_2 = (|I|^2/|\lambda_2|)\|ff''\|_\infty$. Since f, f' and f'' are bounded on I, it follows that c_1 and c_2 are finite constants. Then pick T_0 positive such that $|T| \geq T_0$ implies

$$|T| > c_1 + \sqrt{c_1^2 + 4c_2}. \tag{A 12}$$

Substituting $\phi + T$ for ϕ in (A 11) and applying the method of successive approximations, we consider a sequence of functions ϕ_n that satisfy the initial conditions $\phi_n(a) = 0$, $\phi_n'(a) = u$. Initialize $\phi_0 = 0$ and inductively define

$$\phi_{n+1}(t) = u(t-a) + \frac{\lambda_1}{\lambda_2}(f(a) + f'(a)(t-a) - f(t)) - \frac{1}{\lambda_2}\int_{a}^{t}\int_{a}^{\tau}\frac{f(v)f''(v)}{\phi_n(v) + T}\,\mathrm{d}v\,\mathrm{d}\tau. \tag{A 13}$$

By construction $|\phi_0| = 0 < (|T|/2)$. Assume $|\phi_n| < (|T|/2)$, then $|\phi_n + T| \geq (|T|/2)$. Estimating the r.h.s. of (A 13), it follows from the selection of T in (A 12) that

$$|\phi_{n+1}| \leq c_1 + \frac{2c_2}{|T|} < \frac{|T|}{2}. \tag{A 14}$$

Hence the limit function $\Phi = T + \lim_{n\to\infty}\phi_n$ satisfies (A 11) with $\Phi(\tilde{t}) = T$, $\Phi'(\tilde{t}) = u$ and $|\Phi| \geq (|T|/2)$ on I. ∎

Finally, we include a necessary condition for the existence of a solution of a second-order BVP, the details of which may be found in Hartman (1964).

Proposition A.3. *Suppose $F(t, x, \xi)$ is continuous on the tube domain $E(\rho, R) = \{(t, x, \xi) : 0 \leq t \leq \rho, |x| \leq R, \xi \text{ real}\}$ and that F satisfies the condition that both $F(t, R, 0)$ is non-negative and $F(t, -R, 0)$ is non-positive for all $0 \leq t \leq \rho$. Assume further that there exist positive numbers C, γ with $\gamma < 1/R$ so that $|F| \leq \gamma\xi^2 + C$, then there exists a solution to the BVP*

$$\left.\begin{aligned} x'' &= F(t, x, x') \\ x(0) &= x_0, x(\rho) = x_\rho, |x_0|, |x_\rho| \leq R. \end{aligned}\right\} \tag{A 15}$$

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