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CONSTRUCTIVE APPROXIMATION

A Characterization of the Interpolation Spaces of H^1 and L^∞ on the Line

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Abstract. The Calderón-Mitjagin theorem characterizes all interpolation spaces of the pair of Lebesgue spaces (L^1, L^∞) as the rearrangement-invariant spaces. The results of this paper show that the interpolation spaces of $H^1(\mathbf{R})$ and $L^\infty(\mathbf{R})$ consist of elements whose nontangential maximal functions lie in rearrangement-invariant spaces.

Let X_0 and X_1 be two Banach spaces which are continuously embedded in a common Hausdorff topological vector space. An *admissible operator* for the pair (X_0, X_1) is a linear operator whose domain contains the union of the two spaces and whose restrictions to X_i is a bounded operator on X_i $(i=0, 1)$. A space X is called an *interpolation space* for the pair (X_0, X_1) if each admissible operator T is bounded on X.

For a measurable function φ let φ^* denote its nonincreasing rearrangement (see [4] or [2] for details). In [4] Calderón showed that the interpolation spaces of L^1 and L^∞ are characterized in terms of a quasi-order $<$ (the Hardy-Littlewood-Pólya relation) involving the rearrangements φ^*:

$$\psi < \varphi \;:=\; \int_0^t \psi^*(s)\,ds \le \int_0^t \varphi^*(s)\,ds, \qquad \text{all} \quad t>0. \tag{1}$$

In fact, Calderón showed that a necessary and sufficient condition for $\psi < \varphi$ to hold is that there exists an admissible operator T for (L^1, L^∞), with respective operator norms one, such that $T\varphi = \psi$. The interpolation spaces X are spaces of measurable functions whose norm $\|\cdot\|_X$ satisfies the condition

$$\psi < \varphi \;\Rightarrow\; \|\psi\|_X \le \|\varphi\|_X. \tag{2}$$

The *Peetre K-functional* for (X_0, X_1) is defined by

$$K(f, t; X_0, X_1) := \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\}$$

where the infimum is taken over all decompositions of $f = f_0 + f_1$ with $f_i \in X_i$

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($i=0,1$). Peetre proved that

$$K(\varphi, t; L^1, L^\infty)=\int_0^t \varphi^*(s)\,ds$$

and so (2) may be reformulated in terms of the K-functional for the pair. A pair (X_0, X_1) is called a *Calderón couple* if the condition

$$K(g,t)\le K(f,t), \qquad \textit{all} \quad t>0$$

implies the existence of an admissible operator T (whose norm depends only on the spaces X_0 and X_1) such that

$$Tf=g.$$

Brudnyi and Krugljak [3] have shown that the interpolation spaces of a Calderón couple (X_0, X_1) are exactly the spaces Y (up to equivalent renorming) such that

$$\|f\|_Y=\Phi(K(f,\cdot)), \tag{3}$$

where Φ is an admissible function norm. In fact, it has been proven in [1] that this follows from the "fundamental lemma" of the K-method [6] and a lemma of Lorentz and Shimogaki concerning the quasi-order $<$. We show that a complementary lemma, also due to Lorentz and Shimogaki, plays a critical role in establishing that (H^1, L^∞) is a Calderón couple. In [11] Peter Jones utilized his constructive solutions of $\bar\partial$ equations with Carleson measure data to show that (H^1, H^∞) is a Calderón couple. The general pattern of our proof follows that in [11] but has some noticeable differences and simplifications. This is partly due to the fact that the replacement of H^∞ by L^∞ relaxes the analyticity requirement. In [9] Janson and Jones investigated, among other things, the complex method for the pair (H^1, L^∞) and employ similar techniques to this paper.

Let $\mathbf{R}$ denote the real line and $\mathbf{U}=\{(x,y): y>0\}$, the upper half plane. Let the function f belong to $L^1(\mathbf{R})+L^\infty(\mathbf{R})$. We use the symbol f also to denote the harmonic extension of f to $\mathbf{U}$,

$$f(x,y)=P_y * f(x),$$

where P_y is the Poisson kernel and $*$ denotes convolution on $\mathbf{R}$. For $x\in\mathbf{R}$, denote by $\Gamma_x := \{(t,y)\in\mathbf{U}: |x-t|\le y\}$ the cone with vertex at x. The *nontangential maximal function* of f is defined by $Nf(x):=\sup\{|f(z,y)|: (z,y)\in\Gamma_x\}$. There are several equivalent norms for the Hardy space H^1. We shall use

$$\|f\|_{H^1} := \|Nf\|_{L^1}. \tag{4}$$

An H^1-atom, or in short an *atom*, for an interval I is any function a_I which satisfies

$$\int a_I=0, \qquad |a_I|\le |I|^{-1}\chi_I. \tag{5}$$

Coifman [5] has provided an "atomic" description of H^1:

$$H^1 = \left\{ f : f = \sum_j \lambda_j a_{I_j}, \sum_j |\lambda_j| < \infty \right\},$$

where the a_{I_j} are atoms. Moreover, it was shown that

$$\|f\|_{H^1} \sim \|f\|_{H^1_{at}} := \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_{I_j} \right\}, \tag{6}$$

where $\varphi \sim \psi$ means that there exist positive constants c_1 and c_2 such that $c_1\varphi \leq \psi \leq c_2\varphi$. The last expression in inequality (6) is usually referred to as the atomic H^1 norm. In [13] a simple proof of (6) is presented and it is shown that

$$K(f, t) = K(f, t; H^1, L^\infty) \sim \int_0^t (Nf)^*(s)\, ds, \qquad t > 0. \tag{7}$$

A similar result in terms of the grand maximal operator was obtained earlier in [7], but the estimate (7) is better suited for our purposes.

Theorem 1. *The pair* $(H^1(\mathbf{R}), L^\infty(\mathbf{R}))$ *is a Calderón couple*; *that is, if* $Ng < Nf$, *then there exists a linear operator* T *such that the conditions*

$$\begin{aligned} &\text{(i)} && Tf = g, \\ &\text{(ii)} && \|Th\|_{H^1} \leq c\|h\|_{H^1}, && h \in H^1, \\ &\text{(iii)} && \|Th\|_{L^\infty} \leq c\|h\|_{L^\infty}, && h \in L^\infty, \end{aligned} \tag{8}$$

hold. The constant c *is independent of* f *and* g.

The definition of the H^1 norm (4) shows that H^1 consists of functions f for which Nf belongs to L^1. It is also clear that L^∞ is comprised of functions f such that Nf belongs to L^∞. If X is a rearrangement-invariant space, then $N(X)$ is defined as the space of functions for which the norm

$$\|f\|_{N(X)} := \|Nf\|_X$$

is finite. The question naturally arises as to whether the interpolation spaces for $N(L^1)$ and $N(L^\infty)$ are precisely the spaces $N(X)$. The next result answers this in the affirmative.

Corollary 2. *If* X *is a rearrangement-invariant space, then* $N(X)$ *is an interpolation space for* $(H^1(\mathbf{R}), L^\infty(\mathbf{R}))$. *Conversely, if* Y *is an interpolation space for* $(H^1(\mathbf{R}), L^\infty(\mathbf{R}))$, *then there exists a unique rearrangement-invariant space* X *such that* $Y = N(X)$ *with equivalent norms.*

In order to construct the desired operator T satisfying the properties (8), we first assume that g satisfies the condition

$$\lim_{t\to\infty} (Ng)^*(t) = 0. \tag{9}$$

Let O_n denote the open set $\{Ng > 2^n\}$. Define

$$g_n := \sum_{I \in \mathscr{C}_n} [g - I(g)]\chi_I, \tag{10}$$

where $\mathscr{C}_n$ is the collection of all components of O_n and $I(g)$ denotes the average $|I|^{-1}\int_I g$ of g over the interval I. It is easy to see that

$$\lim_{n\to-\infty} g_n = g \quad \text{almost everywhere} \tag{11}$$

by using the following basic estimate for averages in terms of the nontangential maximal operator (see inequality (3) of [13] and its proof):

$$|I(g)| \le 7 \max_{x\in\partial I} Ng(x). \tag{12}$$

Indeed, since g belongs to $H^1 + L^\infty$ and satisfies (9), the measure of O_n is finite and $O_n \uparrow \mathbf{R}$ as $n \downarrow -\infty$. By inequality (12) it follows that, for $I \in \mathscr{C}_n$, there holds $|I(g)| \le 7 \cdot 2^n$. Hence

$$|g - g_n| = |g\chi_{O_n^c} + \sum_{I\in\mathscr{C}_n} I(g)\chi_I| \le 7\cdot 2^n \tag{13}$$

which converges to 0 as $n \to -\infty$ and so (11) holds.

Our plan is to construct operators $T = T_n$ so that (8) holds with the approximations g_n replacing g and with uniform operator bounds. Using a limiting argument we obtain an operator T to establish similar results for functions g in $H^1 + L^\infty$ which satisfy condition (9). Finally, we remove this last restriction to obtain the general case.

For each integer k define

$$a_k := g_k - g_{k+1}, \tag{14}$$

then it follows by telescoping the sum that

$$g = \sum_{k=-\infty}^{\infty} a_k. \tag{15}$$

The first result indicates the connection of this decomposition with the Peetre K-functional.

Theorem 3. *Suppose that g satisfies* (9) *and the functions a_k are chosen as in* (14), *then*

$$K(g,t) \le \sum_{k=-\infty}^{\infty} \min(\|a_k\|_{H^1}, t\|a_k\|_{L^\infty}) \le cK(g,t), \qquad t>0. \tag{16}$$

Proof. The left-hand inequality follows since $K(\cdot, t)$ is a norm and by the definition of the K-functional. For the right-hand inequality, let I be any interval in $\mathscr{C}_k$. Define the collection of intervals $\mathscr{C}_I := \{J \in \mathscr{C}_{k+1}: J \subset I\}$ and the set $G(I)$ by $G(I) := I \setminus O_{k+1}$. Next set

$$b_I := a_k\chi_I = g\chi_{G(I)} + \sum_{J\in\mathscr{C}_I} J(g)\chi_J - I(g)\chi_I, \tag{17}$$

then b_I satisfies $\int b_I = 0$ and, by inequality (12),

$$|b_I| \le 2^k\chi_{G(I)} + 7\cdot 2^{k+1}\sum_{J\in\mathscr{C}_I}\chi_J + 7\cdot 2^k\chi_I \le 21\cdot 2^k\chi_I.$$

Hence

$$\|a_k\|_{L^\infty} \le 21\cdot 2^k \tag{18}$$

and

$$\|a_k\|_{H^1} \le c\|a_k\|_{H^1_{at}} \le c2^k \sum_{I\in\mathscr{C}_k} |I| \le c2^k|O_k| \tag{19}$$

since $b_I/(21 \cdot 2^k|I|)$ is an H^1 atom. By these two estimates we see that if j is an integer selected so that $2^{j-1} < (Ng)^*(t) \le 2^j$, then

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \min(\|a_k\|_{H^1}, t\|a_k\|_{L^\infty}) &\le c \sum_{k=-\infty}^{\infty} 2^k \min(|O_k|, t)\\ &= c\left\{\sum_{k=j}^{\infty} 2^k|O_k| + t\sum_{k=-\infty}^{j-1} 2^k\right\}\\ &\le c\left\{\sum_{k=j}^{\infty} (2^{k+1}-2^k)|O_k| + t2^j\right\}\\ &\le c\left\{\int_{O_j} Ng + t2^j\right\}\\ &\le c\int_0^t (Ng)^* \le cK(g, t).\end{aligned}$$

In the fourth line we used summation by parts and the fact that $Ng > 2^k$ on the set $O_k \setminus O_{k+1}$. ■

Remark 4. Theorem 3 is actually implicit in the proof given in [13] and may be regarded as an explicit decomposition for Cwikel's version of the *fundamental lemma* in the theory of the real method of interpolation [6]. The proof is included for completeness.

At this stage of the proof we fix n and, for notational convenience, set $\bar{g} := g_n$; that is, we first construct an operator for $\bar{g}$ and will pass to the limit at a later stage. Rather than write this function in the form of the atomic decomposition (see (17))

$$\bar{g} = \sum_{k=n}^{\infty} \sum_{I\in\mathscr{C}_k} b_I, \tag{20}$$

we utilize a stopping time argument to telescope the b_I's locally to scalar multiples of atoms with additional nice properties. We construct recursively a subcollection $\mathscr{C}$ of $\bigcup_n^\infty \mathscr{C}_k$ in the following way. Begin by placing all the intervals from $\mathscr{C}_n$ into $\mathscr{C}$. Next we perform the following *recursive step* for each interval I which has previously been placed in $\mathscr{C}$:

> Define the integer $m(I)$ by $m(I) := \min\{k : |O_k \cap I| \le \frac{1}{2}|I|\}$ and $\mathscr{C}(I)$ to be the collection of components of $O_{m(I)} \cap I$. Add all intervals J from $\mathscr{C}(I)$ to the collection $\mathscr{C}$.

Let $F(I) \subset I$ be defined by

$$F(I) := I \setminus \bigcup_{J\in\mathscr{C}(I)} J = I \setminus O_{m(I)}, \tag{21}$$

then $(Ng)\chi_{F(I)} \le 2^{m(I)}$. Note that the $F(I)$'s are disjoint and

$$O_n = \bigcup_{I \in \mathscr{C}} F(I).$$

In analogy with the decomposition (17) we define

$$g_I := (\bar{g} - \alpha(I))\chi_{F(I)} + \sum_{J \in \mathscr{C}(I)} \alpha(J)\chi_{F(J)}, \tag{22}$$

where

$$\alpha(I) := |F(I)|^{-1} \int_I \bar{g}, \qquad I \in \mathscr{C}. \tag{23}$$

Notice that g_I is supported in I and that

$$\int g_I = \int_{F(I)} \bar{g} - \int_I \bar{g} + \sum_{J \in \mathscr{C}(I)} \int_J \bar{g} = 0. \tag{24}$$

Moreover, the recursive criteria guarantee that

$$|F(I)| \ge |I|/2. \tag{25}$$

Recall that for each $I \in \mathscr{C}_k$ there is an $I_0 \in \mathscr{C}_n$ (the ancestor of I) which contains I and so by inequality (12) we have

$$\begin{aligned} |\alpha(I)| &\le 2|I(\bar{g})| \le 2(|I(g)| + |I_0(g)|) \\ &\le 2(7 \cdot 2^k + 7 \cdot 2^n) \le 28 \cdot 2^k. \end{aligned}$$

It follows that

$$|g_I| \le 28 \cdot 2^{m(I)} \chi_{E(I)} \tag{26}$$

if $E(I)$ is defined as the disjoint union of $F(I)$ with those at the next level

$$E(I) := F(I) \cup \Big(\bigcup_{J \in \mathscr{C}(I)} F(J) \Big). \tag{27}$$

Now $E(I) \subset I$ and at most two of them overlap

$$\sum_{I \in \mathscr{C}} \chi_{E(I)} \le 2 \tag{28}$$

since the $F(I)$'s are disjoint. As a consequence, we may write

$$\bar{g}(x) = \sum_{I \in \mathscr{C}} g_I(x), \tag{29}$$

where for each x there are at most two nonzero terms in the sum. The sum in (29) is our desired decomposition of $\bar{g}$. It follows that

$$\|g_I\|_{H^1} \le c 2^{m(I)} |I| \tag{30}$$

since the function $(28|I|2^{m(I)})^{-1} g_I$ is an H^1-atom by the estimates (24) and (26). Define

$$\tilde{g} := \sum_{I \in \mathscr{C}} 2^{m(I)} \chi_{F(I)}, \tag{31}$$

then, obviously,

$$Ng \leq \tilde{g} \quad \text{on } O_n.$$

Conversely, the next result shows that $\tilde{g}$ is controlled by Ng. In order to establish this result we will need the notion of the "median" of a function $|h|$ over an interval I:

$$m_I(h) := \inf\{\lambda : |\{|h| > \lambda\} \cap I| \leq \tfrac{1}{2}|I|\}$$

and the corresponding maximal operator $\mathscr{m}h$ defined by

$$\mathscr{m}h(x) := \sup_{I \ni x} m_I(h).$$

From the definitions it is clear that

$$\{x\colon \mathscr{m}h(x) > \lambda\} = \{x\colon M(\chi_{\{|h|>\lambda\}})(x) > \tfrac{1}{2}\},$$

where M denotes the Hardy–Littlewood maximal operator. As was pointed out in [10], it follows that

$$|\{\mathscr{m}h > \lambda\}| \leq 3(\tfrac{1}{2})^{-1}\|\chi_{\{|h|>\lambda\}}\|_{L^1} = 6|\{|h| > \lambda\}|,$$

since M is weak type (1, 1). Hence the corresponding decreasing rearrangements must satisfy

(32) $$(\mathscr{m}h)^*(t) \leq h^*(t/6).$$

Proposition 5. *If $\tilde{g}$ is defined by equation* (31), *then*

(33) $$(\tilde{g})^*(t) \leq 2(Ng)^*(t/6), \qquad t > 0.$$

Hence, if $Ng \prec Nf$, then

(34) $$\tilde{g} \prec cNf.$$

Proof. Inequality (33) follows immediately from inequality (32) and the fact that $\tilde{g} \leq 2\mathscr{m}(Ng)$. Relation (34) follows by changing variables. ■

By (34) a variant (see Corollary V.10.5 of [2]) of a decomposition lemma of Lorentz and Shimogaki [12] for the quasi-order $\prec$ implies the existence of pairwise disjoint sets $\{\tilde{E}(I)\}_{I\in\mathscr{C}}$ such that $|\tilde{E}(I)| = |F(I)|$ and

(35) $$2\int_{\tilde{E}(I)} N(f) \geq |F(I)|2^{m(I)}, \qquad I \in \mathscr{C}.$$

There exists a Borel measurable function $\psi\colon \mathbf{R} \to \mathbf{U}$ ($\psi(x) \in \Gamma_x$) such that $|f(\psi(x))| \geq \frac{1}{2}Nf(x)$, so

(36) $$4\int_{\tilde{E}(I)} |f(\psi(s))|\, ds \geq |F(I)|2^{m(I)}, \qquad I \in \mathscr{C}.$$

Define the unimodular function $\omega(x) := \operatorname{sgn} f(\psi(x))$ and the weights $w(I)$ so that

(37) $$w(I)\int_{\tilde{E}(I)} |f(\psi(s))|\, ds = |I|2^{m(I)},$$

then inequalities (36) and (25) show that the $w(I)$ are uniformly bounded with a bound independent of the functions f and $\tilde{g}$.

Lemma 6. *Suppose that $Ng \prec Nf$ and $\tilde{g}$ is defined as the g_n in equation* (10). *If the linear functionals λ_I are defined by*

$$\lambda_I(h) := \frac{\int_{\tilde{E}(I)} h(\psi(s))\omega(s)\, ds}{|I| 2^{m(I)}}, \qquad I \in \mathscr{C}, \tag{38}$$

then the operator T defined by

$$Th(x) := \sum_{I \in \mathscr{C}} w(I)\lambda_I(h) g_I(x) \tag{39}$$

satisfies the conditions (8) *but with $Tf = \tilde{g}$.*

Proof. By equation (37) we have that $w(I)\lambda_I(f) = 1$ and so equation (29) implies that $Tf = \tilde{g}$. By inequality (30), the facts that the $\tilde{E}(I)$'s are disjoint, and $\psi(s)$ belongs to Γ_s it follows that

$$\|Th\|_{H^1} \le c \sum_{I \in \mathscr{C}} \int_{\tilde{E}(I)} |h(\psi(s))|\, ds \le c \int Nh(s)\, ds. \tag{40}$$

Hence T satisfies part (ii) of (8).

Suppose now that h belongs to L^∞, then by inequality (26) and the fact that $|\tilde{E}(I)| = |F(I)| \le |I|$ we have that

$$\begin{aligned} |Th(x)| &\le c \sum_{I \in \mathscr{C}} |\lambda_I(h)| 2^{m(I)} \chi_{E(I)}(x) \\ &\le c\|h\|_{L^\infty} \sum_{I \in \mathscr{C}} \chi_{E(I)} \le c\|h\|_{L^\infty} \end{aligned}$$

holds. The last inequality follows from inequality (28). Hence T satisfies the estimate (iii) of (8) and the lemma is established. ■

Lemma 7. *Suppose now that g satisfies condition* (9) *and $Ng \prec Nf$, then there exists an admissible operator T such that $Tf = g$.*

Proof. We use Lemma 6 to produce a sequence of operators T_n such that $T_n f = g_n$. The T_n's have uniformly bounded operator norms on H^1 and L^∞ which are independent of n, f, and g. Recall that the functions g_n are defined in (10). We employ Calderón's technique [4] to supply the limit operator T with the desired properties (8). Let γ be a Banach limit. Suppose that h belongs to $H^1 + L^\infty$. For each measurable set E of finite measure, let

$$\nu(E) := \gamma\left(\left\{\int_E T_n h\right\}_{n=-1}^{-\infty}\right),$$

then ν is absolutely continuous with respect to Lebesgue measure on **R**. Hence there exists a locally integrable function Th such that

$$\int_E Th = \nu(E)$$

for each set E of finite measure. It follows by the continuity of γ that

$$P_y * Th(t) = \gamma(\{P_y * T_n h(t)\}_{n=-1}^{-\infty}). \tag{41}$$

In particular, equation (41) holds with $(t, y) = \psi(x)$ where ψ is an arbitrary Borel measurable function from $\mathbf{R}$ to $\mathbf{U}$ with $\psi(x) \in \Gamma_x$. So for each set E of finite measure it follows that

$$\int_E N(Th) \leq \gamma\left(\left\{\int_E N(T_n h)\right\}_{n=-1}^{-\infty}\right)$$
$$\leq c \int_0^{|E|} (Nh)^*(s)\, ds,$$

since γ is a positive linear functional on l^∞. Hence

$$N(Th) < cNh.$$

By this last fact, the definition of Th and equation (11) it follows that T satisfies the desired properties. ■

Proof of Theorem 1. Suppose that $f \in H^1 + L^\infty$ and $Ng < Nf$. In view of Lemma 7 we may assume that

$$\lim_{t\to\infty} (Ng)^*(t) =: \alpha > 0,$$

since this is the only case that remains to be proved. Observe that

$$\alpha \leq \lim_{t\to\infty} \frac{\int_0^t (Ng)^*}{t} \leq \lim_{t\to\infty} \frac{\int_0^t (Nf)^*}{t} = \lim_{t\to\infty} (Nf)^*(t),$$

since both integrands are nonincreasing. Hence there exist sets $F_1 \subset F_2 \subset \cdots$ of finite measure increasing to ∞ and a Borel measurable function $\psi: \mathbf{R} \to \mathbf{U}$ such that $\psi(x) \in \Gamma_x$ and

$$|f(\psi(x))| > \tfrac{1}{2}\alpha \qquad \text{for} \quad x \in \bigcup_{j=1}^{\infty} F_j.$$

Let γ be a Banach limit and define the linear functional λ by

$$\lambda(h) := \gamma\left(\left\{|F_j|^{-1} \int_{F_j} h(\psi(s))\omega(s)\, ds\right\}_{j=1}^{\infty}\right),$$

where $\omega(s) := \operatorname{sgn} f(\psi(s))$. Now γ is a Banach limit so it follows that

(42) $$|\lambda(h)| \leq \gamma(\{\|h\|_{L^\infty}\}_{j=1}^\infty) = \|h\|_{L^\infty}$$

and

(43) $$\lambda(f) \geq \tfrac{1}{2}\alpha.$$

Next select the largest integer n_0 such that

$$2^{n_0} \leq \alpha < 2^{n_0+1}.$$

Let g_{n_0} be defined as in (10) and set $b_{n_0} := g - g_{n_0}$. Now g_{n_0} satisfies the hypothesis of Lemma 6 so there exists an admissible operator T_0 such that

$$T_0 f = g_{n_0}. \tag{44}$$

For the portion b_{n_0} of g we define the operator T_1 by

$$T_1 h(x) := \frac{\lambda(h)}{\lambda(f)} b_{n_0}, \qquad h \in H^1 + L^\infty,$$

then $T_1 f = b_{n_0}$. By inequalities (42), (43), and (13) it follows that

$$\|T_1 h\|_{L^\infty} \leq 14 \|h\|_{L^\infty}.$$

Moreover, λ vanishes on H^1. To verify this, note that Nh is integrable and so $|F_j|^{-1} \int_{F_j} Nh \to 0$ as $j \to \infty$. But γ was chosen to take convergent sequences to their limits. Consequently, T_1 is trivially bounded on H^1. The operator $T := T_0 + T_1$ fulfills the statement of the theorem. ∎

Proof of Corollary 2. The fact that $N(X)$ is an interpolation space is straightforward since the estimate (7) holds and

$$K(Tf, t) \leq cK(f, t), \qquad t > 0,$$

for all admissible operators T. For the converse, the Brudnyi–Krugljak theory asserts that Theorem 1 is enough to guarantee that the interpolation spaces Y of (H^1, L^∞) arise as spaces generated by function norms Φ_Y applied to the K-functional:

$$\|f\|_Y \sim \Phi_Y(K(f, \cdot)) \sim \Phi_Y\left(\int_0^{(\cdot)} (Nf)^*(s)\, ds\right),$$

with constants independent of the functions f. Define the norm

$$\|\varphi\|_X := \Phi_Y\left(\int_0^{(\cdot)} \varphi^*(s)\, ds\right)$$

and X as the rearrangement-invariant space of functions for which this norm is finite. It follows that $Y = N(X)$ with equivalent norms. ∎

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