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K-DIVISIBILITY AND A THEOREM OF LORENTZ AND SHIMOGAKI

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ABSTRACT. The Brudnyi-Krugljak theorem on the K-divisibility of Gagliardo couples is derived by elementary means from earlier results of Lorentz-Shimogaki on equimeasurable rearrangements of measurable functions. A slightly stronger form of Calderón's theorem describing the Hardy-Littlewood-Pólya relation in terms of substochastic operators (which itself generalizes the classical Hardy-Littlewood-Pólya result for substochastic matrices) is obtained.

1. Introduction. When (X_0, X_1) is a compatible couple of Banach spaces, the *Peetre K-functional* $K(f,t) = K(f,t;X_0,X_1)$ of an element f in X_0+X_1 is defined by

$$K(f,t) = \inf\{\|g\|_0 + t\|h\|_1\colon\ f = g+h, g \in X_0, h \in X_1\} \tag{1}$$

(cf. [**2**, Chapter III]).

DEFINITION 1. A compatible couple (X_0, X_1) is said to be *K-divisible* if there is a constant c such that, whenever $f \in X_0+X_1$ and Φ_k, $k = 1, 2, \ldots$, are nonnegative, increasing, concave functions on $(0,\infty)$ with $\sum_k \Phi_k(1) < \infty$ and

$$K(f,t) \le \sum_k \Phi_k(t) \qquad (t > 0), \tag{2}$$

then there exist elements f_k, $k = 1, 2, \ldots$, in $X_0 + X_1$ for which $f = \sum_k f_k$ (convergence in $X_0 + X_1$) and

$$K(f_k, t) \le c\Phi_k(t) \qquad (k = 1, 2, \ldots;\ t > 0). \tag{3}$$

Notice from (1) (by taking $g = f$ and $h = 0$) that $K(f,t)$ is bounded from above (by $\|f\|_0$) whenever $f \in X_0$. Similarly (by taking $g = 0$ and $h = f$), $K(f,t)/t$ is bounded from above (by $\|f\|_1$) whenever $f \in X_1$. Couples (X_0, X_1) for which the converse results hold, that is,

$$K(f,t) \le M \quad (t > 0) \quad \Rightarrow \quad f \in X_0 \text{ and } \|f\|_0 \le M \tag{4}$$

and

$$K(f,t)/t \le M \quad (t > 0) \quad \Rightarrow \quad f \in X_1 \text{ and } \|f\|_1 \le M, \tag{5}$$

are called *Gagliardo couples.* Ju. A. Brudnyi and N. Ja. Krugljak [**1**] have established the following result.

THEOREM 1. *Every Gagliardo couple is K-divisible.*

Another proof of Theorem 1 has been given by M. Cwikel [**4**]. The purpose of the present paper is to show that Theorem 1 can be derived rather simply from some of the fundamental properties of rearrangements of measurable functions.

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2. Rearrangements of measurable functions. For our purposes, it will suffice to consider Lebesgue-measurable functions F, G, etc., defined on the positive real axis $(0,\infty)$; the Lebesgue measure of a measurable subset E of $(0,\infty)$ will be denoted by $|E|$. The distribution function $m = m_F$ of such a function F is given by $m(s) = |\{|F| > s\}|$ $(s > 0)$ and the decreasing rearrangement F^* of F by $F^*(t) = \inf\{s\colon\, m(s) \leq t\}$ (cf. [**2**, **5**] for further details).

The Hardy-Littlewood-Pólya relation $\prec$ is defined as follows: we say that $F \prec G$ if, for all $t > 0$,

$$\int_0^t F^*(s)\,ds \leq \int_0^t G^*(s)\,ds. \tag{6}$$

The Hardy-Littlewood-Pólya relation can be characterized as follows in terms of positive substochastic operators (that is, positive operators that are contractions on L^1 and L^∞). This form of the result, due to Calderón [**3**] (see also Mitjagin [**8**]), generalizes the original finite-dimensional result of Hardy, Littlewood and Pólya [**5**, pp. 44–48].

LEMMA 2. *Nonnegative functions F and G satisfy $F \prec G$ if and only if there is a positive substochastic operator T such that $TG = F$ a.e.*

By incorporating a technique developed by Lorentz and Shimogaki [**6**] (cf. also [**7**]), we obtain the following variant of Lemma 2.

LEMMA 3. *Let F and G be nonnegative decreasing functions in $(L^1+L^\infty)(0,\infty)$. Suppose, in addition, that F is a step function and that $F \prec G$. Then there exists a positive substochastic operator T such that $TG = F$ and T is "monotone" in the sense that TH is decreasing whenever H is.*

PROOF. Write

$$F = \sum_{j=1}^n b_j \chi_{(t_{j-1},t_j)},$$

where $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$. Let T_0 be the linear operator defined by

$$T_0 H = \sum_{j=1}^n H_{I_j} \chi_{I_j},$$

where H_{I_j} denotes the average of H over the interval $I_j = (t_{j-1}, t_j)$. It is easy to verify that T_0 is substochastic, "monotone" in the sense described in the statement of the lemma, and satisfies

$$F \prec T_0 G \prec G, \quad |T_0 H| \leq T_0 |H| \quad (\text{for all } H).$$

Furthermore, $T_0 H$ is a step function relative to the intervals I_j. Composition preserves these properties so, in order to prove the lemma, it will suffice to show that there are finitely many such operators T_j such that

$$T = T_n \circ \cdots \circ T_1 \circ T_0 \quad \text{and} \quad TG = F. \tag{7}$$

Suppose $1 \leq j \leq n$ and that $T_1, T_2, \ldots, T_{j-1}$ have been defined this way. We show how T_j is determined. Set $G_k = T_k G_{k-1}$, $k = 1, 2, \ldots, j-1$, and $G_0 = T_0 G$.

By the induction hypothesis, the functions G_k are step functions relative to the intervals I_j and satisfy

$$F \prec G_{j-1} \prec \cdots \prec G_1 \prec G_0 \prec G.$$

Let n_j be the largest integer such that

$$F = G_{j-1} \quad \text{on } (0, t_{n_j}) \tag{8}$$

and let n'_j be the largest integer such that G_{j-1} is constant on $E_j = (t_{n_j}, t_{n'_j})$. If $n'_j = n$, then there exists a nonnegative decreasing function w such that $wG_{j-1} = F$. In this case, (7) holds with $T_jH = wH$ and $T_{j+1}, T_{j+2}, \dots, T_n$ equal to the identity operator. Similarly, if $n_j = n$, then $T_{j-1} \cdots T_0 G = F$ and again (7) holds if we set the remaining operators equal to the identity. In the remaining case where both $n_j < n$ and $n'_j < n$, we construct T_j with the desired properties and in such a way that at least one of n_j or n'_j increases by at least one. By induction, the proof will then be complete since there are at most n steps.

Let $E'_j = (t_{n'_j}, t_{n'_j+1})$ and set $K = |E_j|/|E'_j|$. Let

$$p = \min\{1/(K+1), (a-b)/(a-a')\}, \tag{9}$$

where a is the value of G_{j-1} on E_j, a' is the value of G_{j-1} on E'_j, and b is the largest value of F on E_j, that is, the value of F on (t_{n_j}, t_{n_j+1}). Note that $a' < a$ because of the way in which n'_j was defined. Since $F \prec G_{j-1}$ and (8) holds, we also have $b < a$. It follows that $0 < p < 1$ and $0 < Kp < 1$.

Define the operator T_j by

$$T_jH = \begin{cases} H, & x \notin E_j \cup E'_j, \\ (1-p)H_{E_j} + pH_{E'_j}, & x \in E_j, \\ (1-Kp)H_{E'_j} + KpH_{E_j}, & x \in E'_j. \end{cases} \tag{10}$$

Note that T_j changes the values of H only on the sets E_j and E'_j. Since $p \le 1/(K+1)$, a computation shows that T_jH is decreasing whenever H is. Hence, T_j is "monotone". It is obvious that T_j is linear, positive, and satisfies $|T_jH| \le T_j|H|$. Furthermore, simple calculations show that T_j is a contraction on L^1 and L^∞ so that T_j is substochastic. Hence, by Lemma 2, the function G_j defined by $G_j = T_jG_{j-1}$ satisfies $G_j \prec G_{j-1}$. The condition $p \le (a-b)/(a-a')$, which follows from (9), guarantees that $F \prec G_j$.

Finally, if in (9) we have $p = (a-b)/(a-a')$, then $F = G_j$ on $(0, t_{n_j+1})$ so that $n_{j+1} \ge n_j + 1$. If, on the other hand, we have $p = 1/(K+1)$ in (9), then G_j is constant on $E_j \cup E'_j = E_{j+1}$ so $n'_{j+1} = n'_j + 1$. In either case, at least one of n_j or n'_j increases. With this the proof is complete.

If $F \prec G_1 + G_2$, we shall need to find a representation $F = F_1 + F_2$ of F with $F_k \prec G_k$ $(k = 1, 2)$. Since

$$K(f, t; L^1, L^\infty) = \int_0^t f^*(s)\,ds \qquad (t > 0)$$

(cf. [**2**, p. 184]), such results, which were first established by Lorentz and Shimogaki [**6**, **7**], may be regarded as primitive forms of K-divisibility for the couple (L^1, L^∞). We shall need the following variant in order to establish Theorem 1.

LEMMA 4. *Suppose G_1, G_2 are nonnegative decreasing functions in $(L^1 + L^\infty)(0, \infty)$ and that F is a function of the form*

$$F = \sum_{j=1}^{\infty} b_j \chi_{(0,t_j)} \qquad (t_j > 0, b_j > 0, j = 1, 2, \ldots).$$

If C, C_1, C_2 are nonnegative constants with $C = C_1 + C_2$ and

$$\int_0^t F(s)\,ds \leq C + \int_0^t [G_1(s) + G_2(s)]\,ds, \tag{11}$$

then, for $k = 1, 2$, there exist numbers $\theta_k(j)$ in $[0, 1]$ with

$$\theta_1(j) + \theta_2(j) = 1 \qquad (j = 1, 2, \ldots) \tag{12}$$

and nonnegative decreasing functions

$$F_k = \sum_{j=1}^{\infty} \theta_k(j) b_j \chi_{(0,t_j)}, \tag{13}$$

which satisfy $F_1 + F_2 = F$ and

$$\int_0^t F_k(s)\,ds \leq C_k + \int_0^t G_k(s)\,ds \qquad (k = 1, 2). \tag{14}$$

PROOF. Suppose first that $C = 0$. Let $G = G_1 + G_2$ and, for each $N = 1, 2, \ldots,$ let

$$F^{(N)} = \sum_{j=1}^{N} b_j \chi_{(0,t_j)}. \tag{15}$$

Then, by (11), we have $F^{(N)} \prec F \prec G$ and so, according to Lemma 3, there is a positive, "monotone", substochastic operator S_N such that $S_N G = F^{(N)}$. Denote by A_0 the averaging operator

$$A_0 H = \sum_{j=1}^{N} H_{I_j} \chi_{I_j},$$

where the I_j are the intervals of constancy of $F^{(N)}$. Then $T_N = A_0 \circ S_N$ is a positive, "monotone", substochastic operator such that $T_N H$ is constant on each of the intervals I_j $(j = 1, 2, \ldots, N)$. Consequently, the step-functions $F_k^{(N)} = T_N G_k$ $(k = 1, 2)$ satisfy

$$F_k^{(N)} \prec G_k \quad (k = 1, 2), \qquad F_1^{(N)} + F_2^{(N)} = F^{(N)}, \tag{16}$$

and may be expressed in the form

$$F_k^{(N)} = \sum_{j=1}^{N} \theta_k^{(N)}(j) b_j \chi_{(0,t_j)},$$

where $\theta_1^{(N)}(j) + \theta_2^{(N)}(j) = 1$. Now letting N vary, we use a standard diagonalization argument to obtain a subsequence $\theta_k^{(N_m)}(j)$ which, for each $j = 1, 2, \ldots$ and each $k = 1, 2$, converges to a limit $\theta_k(j)$, say, as $m \to \infty$. Defining F_k by (13), we note that $F_k^{(N_m)} \to F_k$. Hence from (16) and the dominated convergence theorem, we see that (14) holds (with $C_1 = C_2 = 0$).

The proof is similar in the case where $C \neq 0$. With $F^{(N)}$ defined again by (15), we claim that the estimate

$$\int_0^t F^{(N)}(s)\,ds \leq C\min(t/K, 1) + \int_0^t G(s)\,ds \tag{17}$$

holds for all $t > 0$, where $K = \inf\{t_j\colon\ j = 1, 2, \ldots, N\} > 0$. Indeed, (17) holds for all $t \geq K$ by virtue of the hypothesis (11) so we need only verify it on the interval $[0, K]$. As we just remarked, it is true for $t = K$ and it is trivially true for $t = 0$. But $F^{(N)}$ is constant on $[0, K]$ so the left-hand side of (17) is linear there. The right-hand side is concave. Since (17) is true at the endpoints, it is therefore true in all of $[0, K]$, and this establishes the claim made above. Now observe that the minimum in (17) is the integral of a characteristic function so that (17) can be rewritten in the form

$$F^{(N)} \prec [(C_1/K)\chi_{(0,K)} + G_1] + [(C_2/K)\chi_{(0,K)} + G_2].$$

Since this is an estimate of the form (11) with $C = 0$, we can now apply the result established in the first part of the proof. From this, the desired result (14) follows immediately.

3. Proof of Theorem 1. Let (X_0, X_1) be a Gagliardo couple and fix f in $X_0 + X_1$. Let Φ_k, $k = 1, 2, \ldots$, be nonnegative, increasing, concave functions on $(0, \infty)$ satisfying (2). We shall construct elements f_k, $k = 1, 2, \ldots$, with $f = \sum f_k$ for which (3) holds.

As in Cwikel [**4**, Theorem 4], we may represent f in the form $f = \sum_j a_j$ (the series converging in $X_0 + X_1$) in such a way that the estimates

$$K(f, t) \leq \sum_j \min(\|a_j\|_0, t\|a_j\|_1) \leq 18K(f, t) \tag{18}$$

hold for all $t > 0$. Furthermore, the elements a_j can be chosen so that one of the following conditions holds:

(I) $a_j \in X_0 \cap X_1$ for all j;

(II) there is an index Q such that $a_j = 0$ if $j < Q$ and $a_j \in X_0 \cap X_1$ if $j > Q$; the element $a_Q \in X_0$;

(III) there is an index P such that $a_j = 0$ if $j > P$ and $a_j \in X_0 \cap X_1$ if $j < P$; the element $a_P \in X_1$;

(IV) there are indices P and Q with $P > Q$ such that $a_j \in X_0 \cap X_1$ if $Q < j < P$, $a_j = 0$ if $j < Q$ or $j > P$, and $a_Q \in X_0$, $a_P \in X_1$.

We suppose first that (I) holds. Then, with $c = 1/18$, the function

$$\Psi(t) = c\sum_j \min(\|a_j\|_0, t\|a_j\|_1)$$

vanishes at the origin. Set

$$\psi(t) = (d/dt)\Psi(t) = c\sum_j \|a_j\|_1 \chi_{(0,t_j)},$$

where $t_j = \|a_j\|_0/\|a_j\|_1$. Observe from (1) that for all j,

$$K(a_j,t) \le \min(\|a_j\|_0, t\|a_j\|_1) = \int_0^t \|a_j\|_1 \chi_{(0,t_j)}(s)\,ds. \tag{19}$$

If $\phi_k = (d/dt)\Phi_k$ ($k = 1, 2, \ldots$), then, using (18), we may express the hypothesis (2) in the form

$$\int_0^t \psi(s)\,ds \le \sum_k \Phi_k(0) + \int_0^t \left[\phi_1(s) + \sum_{k\ge 2} \phi_k(s)\right] ds. \tag{20}$$

By Lemma 4, there exist numbers $\theta_1(j)$ and $\theta_1'(j)$ in $[0,1]$ with $\theta_1(j)+\theta_1'(j) = 1$ and with the property that if

$$\psi_1 = c\sum_j \theta_1(j)\|a_j\|_1\chi_{(0,t_j)}, \quad \psi_1' = c\sum_j \theta_1'(j)\|a_j\|_1\chi_{(0,t_j)},$$

then

$$\int_0^t \psi_1(s)\,ds \le \Phi_1(0) + \int_0^t \phi_1(s)\,ds = \Phi_1(t) \tag{21}$$

and

$$\int_0^t \psi_1'(s)\,ds \le \sum_{k\ge 2} \Phi_k(0) + \int_0^t \sum_{k\ge 2} \phi_k(s)\,ds. \tag{22}$$

Hence, if we now define

$$f_1 = \sum_j \theta_1(j)a_j, \qquad f_1' = \sum_j \theta_1'(j)a_j$$

so that

$$f = f_1 + f_1', \tag{23}$$

we see from (19) and (21) that

$$cK(f_1,t) \le \int_0^t \psi_1(s)\,ds \le \Phi_1(t), \tag{24}$$

and from (19) and (22) that

$$cK(f_1',t) \le \int_0^t \psi_1'(s)\,ds \le \sum_{k\ge 2} \Phi_k(t). \tag{25}$$

The next step in the inductive procedure is to repeat the argument above with f_1' in place of f and with (22) in place of (20). This produces a decomposition $f_1' = f_2 + f_2'$, where f_2 and f_2' have properties analogous to (24) and (25). At the nth step of the induction, we obtain

$$f_{n-1}' = f_n + f_n', \tag{26}$$

with

$$cK(f_n,t) \le \int_0^t \psi_n(s)\,ds \le \Phi_n(t) \tag{27}$$

and

$$cK(f_n', t) \le \int_0^t \psi_n'(s)\, ds \le \sum_{k\ge n+1} \Phi_k(t). \tag{28}$$

It follows from (23) and (26) that

$$f = \sum_1^n f_k + f_n', \tag{29}$$

so, using (28), we have

$$c\left\|f - \sum_1^n f_k\right\|_{X_0+X_1} = cK(f_n', 1) \le \sum_{k\ge n+1} \Phi_k(1),$$

which tends to 0 as $n \to \infty$. Hence, $f = \sum_k f_k$ in $X_0 + X_1$ and (27) shows that the elements f_k satisfy the desired estimate (3). This completes the proof in case (I).

Suppose now that (II) holds. We may assume that the element a_Q in X_0 does not belong to X_1 and in this case we interpret its norm in X_1 as being infinite. With this interpretation, we define Ψ exactly as before. Note, however, that Ψ no longer vanishes at the origin (in fact, $\Psi(0) = c\|a_Q\|_0$). The derivative ψ is given by the same expression as before, the sum extending over all $j > Q$. The estimate (19) remains valid for all $j > Q$, and when $j = Q$ it is replaced by $K(a_Q, t) \le \|a_Q\|_0$. The constant $\sum \Phi_k(0)$ in (20) is now replaced by $C = \sum \Phi_k(0) - \Psi(0)$, and in order to apply Lemma 4 we have to determine how to represent C as a sum $C = C_1 + C_2$ of positive quantities C_1 and C_2. We do this as follows. For each $k = 1, 2, \ldots$, set

$$\theta_k(Q) = \Phi_k(0) / \sum_j \Phi_j(0).$$

Then $\Phi_k(0) - \theta_k(Q)\Psi(0) \ge 0$ for all k. We set

$$C_1 = \Phi_1(0) - \theta_1(Q)\Psi(0), \qquad C_2 = \sum_{k\ge 2} [\Phi_k(0) - \theta_k(Q)\Psi(0)]$$

and apply Lemma 4. The functions ψ_1 and ψ_1' have exactly the same form as before, the sum in each case extending over all $j > Q$. The estimates (21) and (22) now contain an extra constant term $-\theta_1(Q)\Psi(0)$ and $-\sum_{k\ge 2} \theta_k(Q)\Psi(0)$, respectively, in the right-hand side. However, if f_1 and f_1' are defined as before, with sums extending over all $j \ge Q$, then the conclusions of (23), (24) and (25) are valid as stated. The inductive procedure now follows the same pattern as above.

Similar modifications are necessary in cases (III) and (IV). The details are straightforward and we omit them.

4. Calderón's theorem. We remarked earlier that Calderón's theorem (Lemma 2) is a generalization of an earlier result due to Hardy, Littlewood, and Pólya. The latter result is in fact a finite-dimensional version of Lemma 2 involving substochastic matrices. Calderón reformulated it as a result for substochastic operators acting on step-functions and then obtained the result in its full generality by an approximation process involving Banach limits. It is interesting to note that the same procedure can be applied to Lemma 3 to derive the following generalization of Calderón's theorem. We shall not make use of it here and we omit the proof.

THEOREM 5. *Suppose F and G are nonnegative decreasing functions in $(L^1 + L^\infty)(0, \infty)$ with $F \prec G$.*

(a) *There exists a positive stochastic operator S such that $SG \geq F$ a.e. and with the property that SH is monotone decreasing* (*respectively, increasing*) *whenever H is.*

(b) *There exists a positive substochastic operator T such that $TG = F$ a.e. and with the property that TH is decreasing whenever H is.*

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