INTERPOLATION OF $H^1$ AND $H^\infty$

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The interpolation spaces for $H^1$ and $H^\infty$ are characterized as the Hardy spaces of the interpolation spaces for $L^1$ and $L^\infty$. This description is provided by recent work of Peter Jones on constructive solutions of $\bar{\partial}$ problems and of Brudnyi-Krugljak in the general theory of interpolation.

A Banach space $X$ is called an interpolation space of a Banach couple $(X_0, X_1)$ if each linear operator $T$, whose restriction to $X_i$ is a bounded operator from $X_i$ into itself ($i=0,1$), is also a bounded operator on $X$. In [4] Calderón characterized the interpolation spaces of $L^1$ and $L^\infty$ as the spaces $X$ of measurable functions whose norms satisfy a rearrangement condition; specifically, there must exist a constant $c > 0$ so that

$$f \in X \quad \text{and} \quad g \preceq f \implies \|g\|_X \leq c \|f\|_X.$$  \hfill (1)

Here $g \preceq f$ (the Hardy-Littlewood-Polya preorder) means

$$\int_0^t g^*(s)ds \leq \int_0^t f^*(s)ds, \quad \forall \ t > 0$$

and $g^*$ denotes the decreasing rearrangement of $|g|$ [4]. We call such spaces $X$ rearrangement-invariant function spaces. In this brief note we will provide a description of the interpolation spaces for the Banach couple of Hardy spaces $(H^1, H^\infty)$. For simplicity we work on $\mathbb{R}$ and the upper half plane $\mathbb{R}_+^2$, but similar results hold for $T$ and the disc. The Hardy space

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$H(X)$ of a rearrangement-invariant function space $X$ over $\mathbb{R}$ is defined to be the collection of functions $f$ in $X$ which have analytic extensions into the upper half plane and whose norm is given by

$$||f||_{H(X)} = ||f||_X.$$  

The standard notation of $H^p (\mathbb{R})$ will be used for $H(L^p)$. The main ingredients of the proof are identification of the K-functional for $(H^1, H^\infty)$ by Peter Jones [5], using rather deep constructive results for $\delta^3$ problems, together with recent work in general interpolation theory by Brudnyi and Krugljak [3].

**THEOREM.** A necessary and sufficient condition for a space $Y$ to be an interpolation space for the pair $(H^1, H^\infty)$ is that $Y$ be equal (with equivalent norms) to a Hardy space $H(X)$ for some interpolation space $X$ of the pair $(L^{1},L^{\infty})$ (i.e., for some rearrangement-invariant function space $X$).

Proof. From the proof of Theorem 3 in Jones [5] it follows immediately that the Peetre K-functional (see [3], [7] page 261) for the pair $(H^1, H^\infty)$ can be estimated by

$$c_1 K(f,t) \leq \int_{0}^{t}(Nf)^*(s)ds \leq c_2 K(f,t), \text{ all } t > 0$$

for some fixed positive constants $c_i$ (i=1,2). Here $Nf$ is the nontangential maximal function of $f$ in $R^2_+$; i.e. if $F$ is the harmonic extension of $f$ into $R^2_+$, then $Nf$ is defined by

$$Nf(x) = \text{sup}(|F(t,y)| : (t,y) \in R^2_+, |x-t| \leq y).$$

For our purposes, a slight improvement of (3) is required, namely

$$c_1 K(f,t) \leq \int_{0}^{t} f^*(s)ds \leq c_2 K(f,t), \text{ all } t > 0.$$  

The right hand inequality follows immediately from (3) since $|f| \leq Nf$ a.e. The inequality is evident directly as well since $\int_{0}^{t} f^*$ is a subadditive functional of $f$, $\int_{0}^{t} g^* \leq ||g||_{L^1} = ||g||_{H^1}$ and $\int_{0}^{t} h^* \leq t||h||_{L^\infty} = t||h||_{H^\infty}$.
Hence
\[ \int_0^t f^*(s)ds \leq \inf_{f=g+h} \{ ||g||_{H^1} + t||h||_{H^\infty} \} = K(f,t). \]

For the left hand inequality in (4), let $F$ denote the analytic extension of $f$ into the upper half plane. Factor $F$ as $BG^2$ where $B$ is a Blaschke product and $G$ is a zero-free analytic function in $R^2_+$. Let $g$ be the function on $R$ of boundary values of $G$, then $N_f \leq (Ng)^2$. Hence
\[
\int_0^t (Nf)^*(s)ds \leq \int_0^t (Ng)^*(s)^2ds
\]
\[
\leq c \int_0^t (Mg)^*(s)^2ds
\]
(5)
since $Ng$ is no larger than a constant multiple of the Hardy-Littlewood maximal function $Mg$ (see page 197 of [8]). In addition, Herz's inequality (see, for example,[1]) states that $(Mg)^*(s) \leq \frac{5}{5} \int_0^s g^*(r)dr$, so using the specialized Hardy inequality
\[
\int_0^t [\int_0^s g^*(r)dr/s]^2ds \leq 4 \int_0^t g^*(s)^2ds
\]
(obtained from integration by parts), we obtain
\[
\int_0^t (Mg)^*(s)^2ds \leq c \int_0^t g^*(s)^2ds.
\]
Together with (5) and the fact that $|g|^2 = |f|$ a.e. this shows that for all $t > 0$
\[
\int_0^t (Nf)^*(s)ds \leq c \int_0^t f^*(s)ds
\]
for each $f$ belonging to $H^1 + H^\infty$. This inequality together with Jones' inequality (3) establishes (4).

Suppose now that $X$ is a rearrangement-invariant function space and $H(X)$ is its corresponding Hardy space. If $T$ is a bounded linear operator on both $H^1$ and $H^\infty$, then obviously
\[
K(Tf,t) \leq A K(f,t) \quad \text{all } t > 0
\]
where $A$ is the maximum of the two operator norms. If the estimate (4) is
applied to each side of the last inequality, then Calderón's result (1) shows that $H(X)$ is an interpolation space. Conversely, let $Y$ be an interpolation space for the pair $(H^1, H^\infty)$. Jones has shown [6] that if both $f$ and $g$ belong to $H^1 + H^\infty$ and $g \prec f$, then there exists a linear operator $T$ such that $Tf = g$ and $T$ is bounded on both $H^1$ and $H^\infty$. Applying Corollary 3 of [3], $Y$ is equal to the space $K_\phi(H^1, H^\infty)$ and

$$||f||_Y = \phi(K(f, \cdot))$$

where $\phi$ is the function norm of some interpolation space for the pair $(L^\infty, L^{1/t}_1)$. It follows from (1) that $\phi(\int_0^t g^*(s)ds) = ||g||_X$ is a rearrangement-invariant function norm. Hence by the estimates in (4),

$$c_1 ||f||_Y \leq ||f||_X \leq c_2 ||f||_Y.$$

This result may be rephrased in terms of real Hardy spaces in an obvious way. Recall that $ReH^p$ is the space of functions in $L^p$ whose Hilbert transforms also belong to $L^p$. For $1 < p \leq \infty$, Riesz's theorem shows that $ReH^p$ is $L^p$ with an equivalent norm. In general, for a rearrangement-invariant function space $X$, the real Hardy space of $X$ is defined by

$$ReH(X) = \{f \in X : Hf \in X\}$$

with norm

$$||f||_{ReH(X)} = ||f||_X + ||Hf||_X.$$

Here $Hf$ denotes the Hilbert transform of $f$.

COROLLARY. The K-functional for the pair $(ReH^1, ReH^\infty)$ is equivalent within constants to the expression

$$\int_0^t f^*(s)ds + \int_0^t (Hf)^*(s)ds.$$

The collection of interpolation spaces for this pair are precisely the real
Hardy spaces of the interpolation spaces of \((L^1, L^\infty)\). A necessary and sufficient condition for a rearrangement-invariant Banach function space \(X\) to be an interpolation space for \((\text{Re}H^1, \text{Re}H^\infty)\) is that the Boyd indices of \(X\) satisfy \(0 < \beta_X < \alpha_X < 1\).

Proof. Only the last statement requires verification, but in view of (6), this is precisely the content of [2].

REFERENCES


