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Functional Analysis and Approximation

INTERPOLATION BETWEEN H^1 AND L^∞

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The purpose of this article is to provide a simple proof of the result of N. M. Rivière - Y. Sagher that the real interpolation spaces between H^1 and L^∞ can be identified with the Lorentz L^{pq}-spaces. In contrast to existing proofs, which make heavy use of H^1-structure, the proof given here relies only on the well-known result of E. M. Stein-G. Weiss characterizing the distribution of the Hilbert transform of an arbitrary characteristic function of a set of finite measure, and a simple technique for applying that result due to R. O'Neil-G. Weiss.

1. Introduction

For simplicity only the case of the circle group T will be considered here. When T is equipped with normalized Lebesgue measure, the decreasing rearrangement f^* of a measurable function f on T is the unique nonnegative, decreasing, right-continuous function on the interval (0,1) that is equimeasurable with $|f|$. Recall that the Lorentz space L^{pq} $(1\le p<\infty,\ 1\le q\le\infty)$ consists of all measurable f on T for which

$$(1.1) \qquad ||f||_{pq} = \left(\int_0^\infty [t^{1/p} f^*(t)]^q dt/t \right)^q$$

is finite.

The *periodic Hilbert transform*, or *conjugate-function operator*, H is defined on $L^1(T)$ by the principal-value integral

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$$(Hf)(x) = \tilde{f}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\cot\frac{t}{2}\,dt.$$

The (real) Hardy space $H^1(T)$ consists of those f in L^1 for which $\tilde{f}$ belongs also to L^1: it is a Banach space under the norm

(1.2) $$||f||_{H^1} = ||f||_{L^1} + ||\tilde{f}||_{L^1}.$$

The Peetre K-functional $K(f,t,X_0,X_1)$ for a compatible couple (X_0,X_1) of Banach spaces is defined for every f in X_0+X_1 and every $t > 0$ by

(1.3) $$K(f;t;X_0,X_1) = \inf_{f=f_0+f_1} (||f_0||_{X_0} + t||f_1||_{X_1}).$$

The following result is well-known (cf. [2, p. 184]).

THEOREM 1.1. (J. Peetre) *For each* f *in* $L^1(T)$,

(1.4) $$K(f;t;L^1,L^\infty) = \int_0^t f^*(s)ds = tf^{**}(t) \qquad (t > 0).$$

The real interpolation space $(X_0,X_1)_{\theta,q}$ between X_0 and X_1 consists of those f in X_0+X_1 for which

(1.5) $$||f||_{(X_0,X_1)_{\theta,q}} = \left(\int_0^\infty [t^{-\theta}K(f;t;X_0,X_1)]^q \frac{dt}{t}\right)^{1/q}$$

is finite (cf. [2, p. 167]). Hence, in view of Theorem 1.1 it is a simple matter to use the classical Hardy inequalities to identify the real interpolation spaces between L^1 and L^∞ as follows:

COROLLARY 1.2. *If* $0<\theta<1$, $1\le q\le\infty$, *and* $\theta = 1-1/p$, *then*

(1.6) $$(L^1,L^\infty)_{\theta,q} = L^{pq}$$

with equivalent norms.

The purpose of this note is to establish by simple methods the same result but with L^1 replaced by H^1. The following well-known result (cf. [4, p. 197]) will be crucial.

THEOREM 1.3. (E. M. Stein-G. Weiss). *Let* E *be an arbitrary measurable subset of* T *and let* χ_E *denote its characteristic function. Then*

$$(\tilde{\chi}_E)^*(t) = \frac{1}{\pi} \sinh^{-1}\left(\frac{\sin|E|/2}{\tan \pi t/2}\right) \qquad (0<t<1). \tag{1.7}$$

2. Interpolation between H^1 and L^∞

Let f be a measurable function on T. For each $t>0$, define the truncates f^t and f_t of f by

$$(f^t)(x) = f(x) - f^*(t) \operatorname{sgn} f(x) \tag{2.1}$$

$$f_t = f - f^t. \tag{2.2}$$

The decreasing rearrangements are given by

$$(f^t)^*(s) = \begin{cases} f^*(s) - f^*(t), & 0<s<t, \\ 0, & t\le s<1, \end{cases} \tag{2.3}$$

and

$$(f_t)^*(s) = \begin{cases} f^*(t), & 0<s<t, \\ f^*(s), & t\le s<1, \end{cases} \tag{2.4}$$

so, in particular, for each $t>0$,

$$f^*(s) = (f^t)^*(s) + (f_t)^*(s) \qquad (0<s<1). \tag{2.5}$$

If f belongs to H^1, then since f_t is bounded and hence belongs to H^1, it is clear that $f^t = f - f_t$ is also in H^1. The H^1-norm may be estimated as follows.

LEMMA 2.1. *If* f *belongs to* H^1, *then*

$$\|(f^t)^\sim\|_{H^1} \le c(f^{**})^{**}(t) \qquad (0<t<1), \tag{2.6}$$

where c is a constant independent of f.

PROOF. It follows directly from (2.3) that

$$||f^t||_{L^1} = \int_0^t [f^*(s) - f^*(t)]ds = t[f^{**}(t) - f^*(t)]. \tag{2.7}$$

In order to estimate the L^1-norm of $(f^t)^\sim$ we use a technique employed by R. O'Neil-G. Weiss [4, p. 192]. Let

$$E = \{x: (f^t)^\sim(x) \geq 0\}, \quad F = \{x: (f^t)^\sim(x) < 0\}.$$

Then

$$\begin{aligned} ||(f^t)^\sim||_1 &= \frac{1}{2\pi}\int_E (f^t)^\sim(x)dx - \frac{1}{2\pi}\int_F (f^t)^\sim(x)dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f^t)^\sim(x)\,[\chi_E(x) - \chi_F(x)]dx \\ &= \frac{-1}{2\pi}\int_{-\pi}^{\pi}(f^t)(x)\,[\tilde{\chi}_E - \tilde{\chi}_F](x)dx \\ &\leq \frac{1}{2\chi}\int_0^1 (f^t)^*(s)\,[(\chi_E^\sim)^*(s) + (\chi_F^\sim)^*(s)]ds \end{aligned}$$

Hence, by (2.3), and the monotonicity of $\sinh^{-1}$,

$$\begin{aligned} ||(f^t)^\sim||_1 &\leq \frac{1}{2\pi}\int_0^t (f^*(s) - f^*(t))\,\frac{2}{\pi}\,[\sinh^{-1}(\frac{\sin(|E|/2) + \sin(|F|/2)}{2\tan \pi s/2})]ds \\ &\leq \frac{1}{\pi^2}\int_0^t (f^*(s) - f^*(t))\,\sinh^{-1}(\cot \pi s/2)ds \\ &\leq \frac{1}{\pi^2}\int_0^t f^*(s)\,\sinh^{-1}(\frac{1}{s})ds \end{aligned}$$

An integration by parts gives

$$\begin{aligned} ||(f^t)^\sim||_1 &\leq \frac{1}{\pi^2}\int_0^t sf^{**}(s)\frac{1}{\sqrt{s^2+1}}\frac{ds}{s} \leq \frac{1}{\pi^2}\int_0^t f^{**}(s)ds \\ &= \frac{1}{\pi^2}\,t(f^{**})^{**}(t). \end{aligned}$$

Combining this with (2.7) we obtain

$$||(f^t)^\sim||_{H^1} = ||f^t||_{L^1} + ||(f^t)^\sim||_{L^1}$$

$$\leq ct[f^{**}(t) + (f^{**})^{**}(t)] \leq ct(f^{**})^{**}(t).$$

THEOREM 2.2. *If* $0<\theta<1$, $1\leq q\leq\infty$ *and* $\theta = 1-1/p$, *then*

(2.8) $$(H^1,L^\infty)_{\theta,q} = L^{pq}$$

with equivalent norms.

PROOF. Since the L^1-norm is dominated by the H^1-norm it is clear that $K(f;t;L^1;L^\infty) \leq K(f;t;H^1,L^\infty)$ so by Corollary 1.2,

$$(H^1,L^\infty)_{\theta,q} \subset (L^1,L^\infty)_{\theta,q} = L^{pq}$$

with a continuous embedding. Thus it remains only to show the reverse inclusion.

Fix $t>0$ and write $f = f^t+f_t$ as in (2.1) and (2.2). Then

$$K(f;t;H^1,L^\infty) \leq ||f^t||_{H^1} + t||f_t||_{L^\infty}$$

so from (2.4) and (2.7) we obtain

(2.9) $$K(f;t;H^1,L^\infty) \leq ct(f^{**})^{**}(t) + tf^*(t) \leq ct(f^{**})^{**}(t)$$

Hence from (2.5)

$$||f||_{(H^1,L^\infty)_{\theta,q}} \leq c(\int_0^\infty [t^{1-\theta} (f^{**})^{**}(t)]^q dt/t)^{1/q}$$

whence two applications of Hardy's inequality yields

$$||f||_{(H^1,L^\infty)_{\theta,q}} \leq c(\int_0^\infty [t^{1/p} f^*(t)]^q dt/t)^{1/q} = c||f||_{L^{pq}}$$

This establishes the reverse inclusion $L^{pq} \subset (H^1, L^\infty)_{\theta,q}$ and hence completes the proof.

Rivière and Sagher [5] were the first to establish (2.8). Shortly thereafter Fefferman, Rivière, and Sagher [3] discovered the K-functional for H^p and L^∞ within constants for $0<p<\infty$ by making heavy use of the then newly developed Fefferman-Stein H^p theory. In [1] equation (2.9) was established using $L \log L$ estimates and was incorporated into the framework of weak type inequalities. The proof presented in this paper, although simple, does not extend to $0<p<1$, but does have an easy generalization to $H^1(R^n)$ by using the analogous estimate for Riesz transforms that we stated in Theorem 1.3 [4, p. 193-196].

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