INTERPOLATION BETWEEN $H^1$ AND $L^\infty$

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The purpose of this article is to provide a simple proof of the result of N. M. Rivièrè - Y. Sagher that the real interpolation spaces between $H^1$ and $L^\infty$ can be identified with the Lorentz $L^{p,q}$-spaces. In contrast to existing proofs, which make heavy use of $H^1$-structure, the proof given here relies only on the well-known result of E. M. Stein-G. Weiss characterizing the distribution of the Hilbert transform of an arbitrary characteristic function of a set of finite measure, and a simple technique for applying that result due to R. O'Neil-G. Weiss.

1. Introduction

For simplicity only the case of the circle group $T$ will be considered here. When $T$ is equipped with normalized Lebesgue measure, the decreasing rearrangement $f^*$ of a measurable function $f$ on $T$ is the unique nonnegative, decreasing, right-continuous function on the interval $(0,1)$ that is equi-measurable with $|f|$. Recall that the Lorentz space $L^{p,q}$ $(1 \leq p < \infty$, $1 \leq q \leq \infty)$ consists of all measurable $f$ on $T$ for which

\begin{equation}
||f||_{pq} = \left( \int_0^\infty [t^{1/p}f^*(t)]^q dt / t^q \right)^{1/q}
\end{equation}

is finite.

The periodic Hilbert transform, or conjugate function operator, $H$ is defined on $L^1(T)$ by the principal-value integral

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\[(Hf)(x) = \tilde{f}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \cot \frac{\pi t}{2} \, dt.\]

The (real) Hardy space \(H^1(T)\) consists of those \(f\) in \(L^1\) for which \(\tilde{f}\) belongs also to \(L^1\): it is a Banach space under the norm

\[
||f||_{H^1} = ||f||_{L^1} + ||\tilde{f}||_{L^1}.
\]

The Peetre \(K\)-functional \(K(f,t;X_0,X_1)\) for a compatible couple \((X_0,X_1)\) of Banach spaces is defined for every \(f\) in \(X_0 + X_1\) and every \(t > 0\) by

\[
K(f; t; X_0, X_1) = \inf_{f = f_0 + f_1} \left( ||f_0||_{X_0} + t ||f_1||_{X_1} \right).
\]

The following result is well-known (cf. [2, p. 184]).

**Theorem 1.1.** (J. Peetre) For each \(f\) in \(L^1(T)\),

\[
K(f; t; L^1, L^\infty) = \int_0^t f^*(s) \, ds = tf^{**}(t) \quad (t > 0).
\]

The real interpolation space \((X_0,X_1)_{\theta,q}\) between \(X_0\) and \(X_1\) consists of those \(f\) in \(X_0 + X_1\) for which

\[
||f||_{(X_0,X_1)_{\theta,q}} = \left( \int_0^\infty \left[ \int_0^\theta K(f; t; X_0, X_1) \right]^{q} \frac{dt}{t} \right)^{1/q}
\]

is finite (cf. [2, p. 167]). Hence, in view of Theorem 1.1 it is a simple matter to use the classical Hardy inequalities to identify the real interpolation spaces between \(L^1\) and \(L^\infty\) as follows:

**Corollary 1.2.** If \(0 < \theta < 1\), \(1 \leq q \leq \infty\), and \(\theta = 1 - 1/p\), then

\[
(L^1, L^\infty)_{\theta,q} = L^{pq}
\]

with equivalent norms.

The purpose of this note is to establish by simple methods the same result but with \(L^1\) replaced by \(H^1\). The following well-known result
(cf. [4, p. 197]) will be crucial.

**THEOREM 1.3.** (E. M. Stein-G. Weiss). Let \( E \) be an arbitrary measurable subset of \( T \) and let \( \chi_E \) denote its characteristic function. Then

\[
(\chi_E^\ast)(t) = \frac{1}{\pi} \sinh^{-1} \left( \frac{\sin|E|/2}{\tan \pi t/2} \right) \quad (0 < t < 1).
\]

2. **Interpolation between \( H^1 \) and \( L^\infty \)**

Let \( f \) be a measurable function on \( T \). For each \( t > 0 \), define the truncates \( f^\ast_t \) and \( f_{-t} \) of \( f \) by

\[
(f^\ast_t)(x) = f(x) - f^\ast(t) \text{ sgn } f(x)
\]

\[
f_{-t} = f - f^\ast_t.
\]

The decreasing rearrangements are given by

\[
(f^\ast_t)^\ast(s) = \begin{cases} f^\ast(s) - f^\ast(t), & 0 < s < t, \\ 0, & t \leq s < 1, \end{cases}
\]

and

\[
(f_{-t})^\ast(s) = \begin{cases} f^\ast(t), & 0 < s < t, \\ f^\ast(s), & t \leq s < 1, \end{cases}
\]

so, in particular, for each \( t > 0 \),

\[
f^\ast(s) = (f^\ast_t)^\ast(s) + (f_{-t})^\ast(s) \quad (0 < s < 1).
\]

If \( f \) belongs to \( H^1 \), then since \( f_t \) is bounded and hence belongs to \( H^1 \), it is clear that \( f^\ast_t = f - f_{-t} \) is also in \( H^1 \). The \( H^1 \)-norm may be estimated as follows.

**LEMMA 2.1.** If \( f \) belongs to \( H^1 \), then

\[
\| (f^\ast_t)^\ast \|_{H^1} \leq c(f^\ast)^\ast(f^\ast)(t) \quad (0 < t < 1),
\]
where \( c \) is a constant independent of \( f \).

**PROOF.** It follows directly from (2.3) that

\[
(2.7) \quad \left| \left| f^t \right| \right|_1 \leq \int_0^t \left[ f^*(s) - f^*(t) \right] ds = t[f^{***(t)} - f^{**}(t)].
\]

In order to estimate the \( L^1 \)-norm of \( f^t \) we use a technique employed by R. O'Neil-G. Weiss [4, p. 192]. Let

\[
E = \{ x: (f^t)^\sim(x) \geq 0 \}, \quad F = \{ x: (f^t)^\sim(x) < 0 \}.
\]

Then

\[
\left| \left| (f^t)^\sim \right| \right|_1 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} (f^t)^\sim(x) dx - \frac{1}{2\pi} \int_{-\pi}^{\pi} (f^t)^\sim(x) dx
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f^t)^\sim(x) \left[ x_E^*(x) - x_F^*(x) \right] dx
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f^t)^\sim(x) \left[ x_E^\sim - x_F^\sim \right] (x) dx
\]

\[
\leq \frac{1}{2\pi} \int_0^t (f^t)^*(s) \left[ (x_E^\sim)^*(s) + (x_F^\sim)^*(s) \right] ds
\]

Hence, by (2.3), and the monotonicity of \( \sinh^{-1} \),

\[
\left| \left| (f^t)^\sim \right| \right|_1 \leq \frac{1}{2\pi} \int_0^t (f^*(s) - f^*(t)) \frac{2}{\pi} \left[ \sinh^{-1}\left( \frac{\sin(\|E\|/2) + \sin(\|F\|/2)}{2 \tan \pi s/2} \right) \right] ds
\]

\[
\leq \frac{1}{\pi} \int_0^t (f^*(s) - f^*(t)) \sinh^{-1}(\cot \pi s/2) ds
\]

\[
\leq \frac{1}{\pi} \int_0^t f^*(s) \sinh^{-1}(\frac{1}{s}) ds
\]

An integration by parts gives

\[
\left| \left| (f^t)^\sim \right| \right|_1 \leq \frac{1}{\pi} \int_0^t sf^{**}(s) \frac{1}{\sqrt{s^2 + 1}} ds \leq \frac{1}{\pi} \int_0^t f^{**}(s) ds
\]

\[
= \frac{1}{\pi} \pi f^{**}(t).
\]

Combining this with (2.7) we obtain
\[
\| (f^t)^{\sim} \|_{L^1} = \| f^t \|_{L^1} + \| (f^t)^{\sim} \|_{L^1}
\leq c \| f^{\star \star} (t) + (f^{\star \star})^{\star \star} (t) \| \leq c (f^{\star \star})^{\star \star} (t).
\]

**Theorem 2.2.** If 0 < \( \theta < 1 \), 1 \( \leq q \leq \infty \) and \( \theta = 1 - 1/p \), then

\[
(2.8) \quad (H^1, L^\infty)^\theta, q = L^{pq}
\]

with equivalent norms.

**Proof.** Since the \( L^1 \)-norm is dominated by the \( H^1 \)-norm it is clear that

\[
K(f; t; L^1, L^\infty) \leq K(f; t; H^1, L^\infty)
\]

so by Corollary 1.2,

\[
(2.4) \quad (H^1, L^\infty)^\theta, q \subseteq (L^1, L^\infty)^\theta, q = L^{pq}
\]

with a continuous embedding. Thus it remains only to show the reverse inclusion.

Fix \( t > 0 \) and write \( f = f^t + f_t \) as in (2.1) and (2.2). Then

\[
K(f; t; H^1, L^\infty) \leq \| f^t \|_{H^1} + t \| f_t \|_{L^\infty}
\]

so from (2.4) and (2.7) we obtain

\[
K(f; t; H^1, L^\infty) \leq c (f^{\star \star})^{\star \star} (t) + tf^{\star} (t)
\]

\[
(2.9)
\]

Hence from (2.5)

\[
\| f^t \|_{(H^1, L^\infty)^\theta, q} \leq c \int_0^\infty [t^{1-\theta} (f^{\star \star})^{\star \star} (t)]^q dt / t \}
\]

whence two applications of Hardy's inequality yields

\[
\| f^t \|_{(H^1, L^\infty)^\theta, q} \leq c \int_0^\infty [t^{1/p} f^{\star} (t)]^q dt / t \}
\]

\[
= c \| f \|_{L^{pq}}
\]
This establishes the reverse inclusion \( L^{pq} \subseteq (H^1, L^\infty)_\Theta, q \) and hence completes the proof.

Riviè re and Sagher [5] were the first to establish (2.8). Shortly thereafter Fefferman, Riviè re, and Sagher [3] discovered the K-functional for \( H^p \) and \( L^\infty \) within constants for \( 0 < p < \infty \) by making heavy use of the then newly developed Fefferman–Stein \( H^p \) theory. In [1] equation (2.9) was established using \( L \log L \) estimates and was incorporated into the framework of weak type inequalities. The proof presented in this paper, although simple, does not extend to \( 0 < p < 1 \), but does have an easy generalization to \( H^1(R^n) \) by using the analogous estimate for Riesz transforms that we stated in Theorem 1.3 [4, p. 193-196].

REFERENCES


