Weak-$L^\infty$ and BMO

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Dedicated to Professor George G. Lorentz on the occasion of his seventieth birthday

1. Introduction

The Marcinkiewicz space weak-$L^p$ properly contains $L^p$ when $0 < p < \infty$ but it coincides with $L^\infty$ when $p = \infty$. Consequently, the Marcinkiewicz interpolation theorem does not directly apply to operators that are unbounded on $L^\infty$. The main purpose of this paper is to construct a rearrangement-invariant space $W$ that will play the role of “weak-$L^\infty$”, in the sense that it contains $L^\infty$ and possesses the appropriate interpolation properties. The construction, which is motivated by elementary considerations in the Lions-Peetre real interpolation method, is valid for general measure spaces. However, if the underlying measure space is a cube in $\mathbb{R}^n$, then $W$ has an alternative characterization in terms of the space BMO of functions of bounded mean oscillation.

The space $W$ consists of those measurable functions $f$ for which $f^{**} - f^*$ is bounded (where $f^*$ is the decreasing rearrangement of $f$ and $f^{**(t)} = t^{-1}\int_0^t f^*(s)ds$). Although no explicit use will be made of the fact, it is perhaps of some interest to note that the space $W$ so-defined arises via the real interpolation method from the pair $(L^\infty, L^1)$ in exactly the same way that the space weak-$L^1$ arises from the reversed pair $(L^1, L^\infty)$. This and other properties of $W$ are developed in Section 2. In particular, a Marcinkiewicz-type interpolation theorem is established for $W$ and it is shown that this result gives a direct proof of the $L^p$-boundedness of the Hilbert transform and related singular integral operators for all values of $p$ with $1 < p < \infty$. With these properties, and the fact that $W$ can be realized as a limit of the familiar spaces weak-$L^p$ as $p \to \infty$, the space $W$ may justifiably be referred to as weak-$L^\infty$.

The relationship between weak-$L^\infty$ and BMO is established in Section 3. A covering argument is used to relate the oscillation of a function $f$ to that...
of its decreasing rearrangement \(f^*\), and thereby to establish the main result that weak-\(L^{\infty}(Q)\), where \(Q\) is a cube in \(\mathbb{R}^n\), is precisely the rearrangement-invariant hull of BMO(\(Q\)).

In the final section the Hardy-Littlewood maximal operator is shown to be bounded from \(W\) into \(W\) and from BMO into BMO.

2. The space weak-\(L^{\infty}\)

The Peetre \(K\)-functional for the pair \((L^1, L^{\infty})\), with respect to an arbitrary \(\sigma\)-finite measure space \((X, \mu)\), can be explicitly identified as follows:

\[
K(f, t; L^1, L^{\infty}) = \int_0^t f^*(s) \, ds = tf^{**}(t) \quad (t > 0)
\]

(cf. [2, p. 184]). The norm in the Marcinkiewicz space weak-\(L^1\) is therefore given in terms of the \(K\)-functional by

\[
\|f\|_{\text{weak-}L^1} = \sup_{t>0} tf^*(t) = \sup_{t>0} t \frac{d}{dt} K(f, t; L^1, L^{\infty}).
\]

If the roles of \(L^1\) and \(L^{\infty}\) are now reversed, then a simple computation, together with the identity \(K(f, t; L^{\infty}, L^1) = tK(f, t^{-1}; L^1, L^{\infty})\), shows that the functional corresponding to that on the right of (2.1) is simply

\[
\sup_{t>0}[f^{**}(t) - f^*(t)].
\]

**Definition 2.1.** Let \(W = W(X)\) denote the set of \(\mu\)-measurable functions \(f\) on \(X\) for which \(f^*(t)\) is finite for all \(t > 0\) and for which \(f^{**}(t) - f^*(t)\) is a bounded function of \(t\). Let

\[
\|f\|_W = \sup_{t>0}[f^{**}(t) - f^*(t)] \quad (f \in W).
\]

It is clear that \(W\) contains \(L^{\infty}\), and the containment is proper on the interval \((0, 1)\) (or any nonatomic measure space) since \(\log(1/t)\), for example, belongs to \(W(0, 1)\) but not to \(L^{\infty}(0, 1)\). This logarithmic rate of growth for \(f^*\) at the origin is in fact the maximum attainable for any \(f\) in \(W\). This follows at once from the elementary identity

\[
f^{**}(t) - f^{**}(s) = \int_s^t [f^{**}(u) - f^*(u)] \frac{du}{u} \quad (0 < t \leq s < \infty)
\]

by putting \(s = 1\) and using (2.2) to estimate the integrand. But such a growth condition does not characterize \(W\), as easy examples show. The fact is that membership in \(W\) depends not on the growth of \(f^*\) or \(f^{**}\) but rather on the growth of the *derivative* of \(f^{**}\). In fact, a simple computation gives

\[
f^{**}(t) - f^*(t) = -t \frac{d}{dt} (f^{**}(t))
\]
at each point of differentiability of $f^{**}$, that is, at each point of continuity of $f^\ast$. It should also be pointed out that $W$ is not a linear space: there are in fact nonnegative functions in $W$ whose sum is not in $W$. There are also functions $f$ in $W$ such that neither $f_+$ nor $f_-$ belongs to $W$.

When $1 < p < \infty$, it follows from (2.3) (with $s = \infty$) that the functional
\[
\left( \int_0^\infty \left( t^{1/p} [f^{**}(t) - f^\ast(t)] \right)^q \frac{dt}{t} \right)^{1/q} \quad (0 < q \leq \infty)
\]
is finite if and only if $f$ belongs to the Lorentz space $L^{p^\prime}$. With $q = 1$, this expression converges to $\|f\|_{L^\infty}$ as $p \to \infty$. Thus $L^{\infty}$ may be regarded in this way as the limit of the Lorentz spaces $L^{p^\prime}$. By the same token the space $W$ is the limit as $p \to \infty$ of the Lorentz spaces $L^{p^\prime} = \text{weak-}L^p$. This suggests the following definition.

Recall [10, p. 184] that a sublinear operator $T$ is of weak type $(1, 1)$ if it is a bounded map from $L^1$ into weak-$L^1$:
\[(2.4) \quad \sup_{t > 0} t(Tf^\ast)(t) \leq c \int_0^\infty f^\ast(t)dt \quad (f \in L^1).\]
By analogy, $T$ will be said to be of weak type $(\infty, \infty)$ if it is a bounded map from $L^{\infty}$ into $W$:
\[(2.5) \quad \sup_{t > 0} [(Tf)^{**}(t) - (Tf^\ast)(t)] \leq c \sup_{t > 0} f^\ast(t) \quad (f \in L^{\infty}).\]
Our interpolation theorem will merely require that (2.4) and (2.5) hold for characteristic functions. Hence, in accordance with the Stein-Weiss terminology [10, p. 197], a sublinear operator $T$ will be of restricted weak type $(1, 1)$ (respectively, restricted weak type $(\infty, \infty)$) if its domain contains all simple functions and if (2.4) (respectively, (2.5)) holds for all characteristic functions $f = \chi_E$ of sets $E$ of finite measure. The following interpolation theorem is best formulated in terms of the Calderón maximal operator $S$ [3, p. 288]:
\[(Sf)(t) = \frac{1}{t} \int_0^t f(u)du + \int_t^\infty f(u)\frac{du}{u} \quad (t > 0).\]

**Theorem 2.2.** Let $T$ be a sublinear operator of restricted weak types $(1, 1)$ and $(\infty, \infty)$. Then, for all simple functions $f$,
\[(2.6) \quad (Tf)^{**}(t) \leq cS(f^{**})(t) \quad (t > 0)\]
and
\[(2.7) \quad \|Tf\|_{L^p} \leq c_p \|f\|_{L^p} \quad (1 < p < \infty),\]
where $c$ depends only on $T$, and $c_p$ only on $p$ and $T$. In particular, if $T$ is linear, then $T$ has a unique extension to a bounded linear operator on $L^p$ ($1 < p < \infty$).
Proof. Let $E$ be any $\mu$-measurable subset of $X$ with $0 < s = \mu(E) < \infty$. Let $\chi$ denote the characteristic function of $E$ and let $g = T\chi$. Then the hypotheses on $T$ (cf. (2.4) and (2.5)) give

$$tg^*(t) \leq cs \quad (t > 0)$$

and

$$g^{**}(t) - g^*(t) \leq c \quad (t > 0),$$

where $c$ is a constant depending only on $T$. These estimates may be combined to give

$$g^*(t) \leq 2c\left\{\left(\frac{s}{t} \wedge 1\right) + \log^+(\frac{s}{t})\right\} \quad (t > 0).$$

This follows at once from (2.8) if $t \geq s$. In the remaining case where $0 < t < s$, the estimate (2.9) may be used to estimate the integrand in (2.3) (applied to $g$) to give $g^{**}(t) \leq g^{**}(s) + c \log(s/t)$, and this yields (2.10) since successive applications of (2.9) and (2.8) show that $g^{**}(s) \leq g^*(s) + c \leq 2c$.

The right-hand side of (2.10) is precisely $2cS(\chi^*)(t)$, where $S$ is the Calderón operator. Hence (2.10) may be written in the form

$$(T\chi)^*(t) \leq 2cS(\chi^*)(t) \quad (t > 0).$$

An integration of both sides and some further computation now yield the more desirable form

$$(T\chi)^{**}(t) \leq 2cS(\chi^{**})(t) \quad (t > 0),$$

the point being that the operation $f \rightarrow f^{**}$ is subadditive whereas $f \rightarrow f^*$ is not. This, together with the sublinearity of $T$, enables us, with standard arguments (cf. [3, pp. 286–287]), to pass from the estimate (2.11) for characteristic functions to the desired estimate (2.6) for all simple functions. The remaining assertions are routine consequences of this one.

The Hilbert transform $H$ may be interpolated directly by the previous theorem. All that is needed is the Stein-Weiss estimate [10, p. 240]

$$(H\chi_E)^*(t) = \frac{1}{\pi} \sinh^{-1}\left(\frac{2|E|}{t}\right) \quad (t > 0),$$

valid for any subset $E$ of $(-\infty, \infty)$ with finite measure $|E|$. It follows at once from this identity that $H$ is of restricted weak types $(1, 1)$ and $(\infty, \infty)$, and hence that $H$ may be interpolated by Theorem 2.2. The interpolation theorem applies also to the maximal Hilbert transform and, more generally, to the maximal operators associated with arbitrary Calderón-Zygmund singular integrals (cf. [9, p. 35]).

It is worth pointing out that Herz [5] has an interpolation theorem
which is somewhat loosely related to ours. The functional $f^{**} - f^*$ is implicit in the proof and it plays a prominent role in some of Herz’ applications to martingales. Our interpolation theorem may also be compared with a result of N. M. Riviére [6], to the effect that if $T$ is of weak type $(1, 1)$ and maps $L^\infty$ into BMO, then $T$ is bounded on every $L^p$ with $1 < p < \infty$. In view of Theorem 3.1 of the next section, this result is contained in ours, at least when the underlying measure space is a cube in $\mathbb{R}^n$.

3. Weak-$L^\infty$ and BMO

In this section the underlying measure space will be a fixed cube $Q$ (with sides parallel to the coordinate axes) in $\mathbb{R}^n$ with Lebesgue measure. For each integrable function $f$ on $Q$, the sharp function of $f$ relative to $Q$ is defined by

$$(3.1) \quad f^*_Q(x) = \sup_{Q' \ni x \subset Q} \frac{1}{|Q'|} \int_{Q'} |f'(y) - f_Q| \, dy$$

where $f_Q = 1/|Q'| \int_{Q'} f(y) \, dy$ and the supremum is taken over all cubes $Q'$ that contain $x$ and are contained in $Q$. If $f^*_Q$ is a bounded function of $x$, then $f$ is said to belong to BMO$(Q)$. The norm is given by

$$(3.2) \quad \|f\|_{\text{BMO}(Q)} = \sup_{x \in Q} f^*_Q(x) .$$

It is well-known that BMO can serve as a useful substitute for $L^\infty$ (cf. [4], [6], [7], [8], [11]). The next theorem shows that BMO for a cube $Q$ is intimately connected with $W(Q)$.

**Theorem 3.1.** (a) If $f$ belongs to $L^1(Q)$, then

$$(3.3) \quad f^{**}(t) - f^*(t) \leq c(f^*_Q)^*(t) \quad (0 < t < \frac{1}{6} |Q|),$$

where $c$ is a constant depending only on $n$.

(b) The space $W(Q)$ is the rearrangement-invariant hull of BMO$(Q)$ in the sense that an integrable function $f$ belongs to $W(Q)$ if and only if $f$ is equimeasurable with some function $g$ in BMO$(Q)$.

The following covering lemma, which is a variant of Lemma 1.1 in [1], will be needed. The proof is similar so we omit it.

**Lemma 3.2.** Let $\mathcal{O}$ be a relatively open subset of $Q$ such that $|\mathcal{O}| < (1/2)|Q|$. Then there is a family of cubes $Q_j$ ($j = 1, 2, \cdots$) with pairwise disjoint interiors such that

(i) $|\mathcal{O} \cap Q_j| \leq 2^{-i} |Q_j| < |\mathcal{O}^* \cap Q_j|$ \hspace{1cm} $(j = 1, 2, \cdots)$;

(ii) $\mathcal{O} \subset \bigcup_{j=1}^\infty Q_j \subset Q$;

(iii) $|\mathcal{O}| \leq \sum_{j=1}^{\infty} |Q_j| \leq 2^{k+1} |\mathcal{O}|$. 

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Proof of Theorem 3.1. Since \(|f|_c \leq 2f_0^\prime\), it is enough to establish (3.3) for nonnegative \(f\). In that case, fix \(t\) with \(0 < t < (1/6)|Q|\) and let
\[ E = \{x \in Q : f(x) > f^\prime(t)\}, \quad F = \{x \in Q : f^\prime_0(x) > (f^\prime_0)^\prime(t)\}. \]
Then \(|E \cup F| \leq 2t\) so there is a relatively open subset \(\emptyset\) of \(Q\) with \(|\emptyset| \leq 3t\) and \(E \cup F \subset \emptyset \subset Q\). In particular \(|\emptyset| \leq (1/2)|Q|\) so by Lemma 3.2 there is a covering \(\{Q_j\}_{j=1}^\infty\) of \(\emptyset\) satisfying conditions (i), (ii), and (iii) above. Now
\[
\begin{align*}
t\{f^{**}(t) - f^*(t)\} &= \int_Q \{f(x) - f^*(t)\}dx = \sum_{j=1}^\infty \int_{Q \cap Q_j} \{f(x) - f^*(t)\}dx \\
&\leq \sum_j \int_{Q_j} |f(x) - f_{Q_j}^\prime|dx + \sum_j E \cap Q_j \{f_{Q_j} - f^*(t)\} \\
&= A + B, \text{ say.}
\end{align*}
\]
If \(\Sigma^\prime\) denotes the sum over those indices \(j\) for which \(f_{Q_j}^\prime > f^*(t)\), then
\[
B \leq \Sigma^\prime |E \cap Q_j| \{f_{Q_j} - f^*(t)\} \leq \Sigma^\prime |\emptyset \cap Q_j| \{f_{Q_j} - f^*(t)\}.
\]
Hence, by (i),
\[
B \leq \Sigma^\prime \int_{\emptyset \cap Q_j} \{f_{Q_j} - f^*(t)\}dx \leq \Sigma^\prime \int_{Q_j} |f_{Q_j} - f(x)|dx \leq A,
\]
where the middle inequality holds because \(f(u) \leq f^*(t)\) on \(\emptyset^\prime\). This, together with the preceding estimate, gives
\[
(3.4) \quad t\{f^{**}(t) - f^*(t)\} \leq 2A.
\]
Now observe from (i) that each \(Q_j\) meets \(F^\prime\) in at least one point, say \(x_j\). Then \(f_{Q_j}^\prime(x_j) \leq (f_{Q_j}^\prime)^\prime(t)\) because of the way \(F\) is defined, and so
\[
A = \sum_j |Q_j| \left\{ \frac{1}{|Q_j|} \int_{Q_j} |f(x) - f_{Q_j}^\prime| dx \right\} \leq \sum_j |Q_j| f_{Q_j}^\prime(x_j) \leq \sum_j |Q_j| (f_{Q_j}^\prime)^\prime(t).
\]
Hence, by (iii),
\[
A \leq 2^{n+1} |\emptyset| (f_{Q_j}^\prime)^\prime(t) \leq 2^{n+1} (3t)(f_{Q_j}^\prime)^\prime(t),
\]
and this together with (3.4) establishes (3.3).

For part (b), note first that if \(t \geq (1/6)|Q|\), then
\[
f^{**}(t) - f^*(t) \leq f^{**} \left( \frac{1}{6} |Q| \right) \leq 6 f^{**}(|Q|) = \frac{6}{|Q|} \int_Q |f(x)|dx.
\]
The inequality (3.3) may be used to estimate \(f^{**} - f^*\) in the case \(t < (1/6)|Q|\), so together these estimates give
\[
(3.5) \quad \|f\|_{W(Q)} \leq c \left( \|f\|_{BMO(Q)} + \frac{1}{|Q|} \int_Q |f(x)|dx \right).
\]
This shows that \(BMO(Q)\) is contained in \(W(Q)\) and hence, since \(W(Q)\) is rearrangement-invariant, that every function \(f\) equimeasurable to a
BMO(Q)-function $g$ must lie in $W(Q)$.

It will suffice to prove the converse for the unit cube $Q = I^n$ (where $I = [0, 1]$) since a linear change of variables reduces the general case to this one. But then if $f \in W(I^n)$, the function

$$g(x) = f^*(x_i) \quad (x = (x_1, x_2, \ldots, x_n) \in I^n)$$

is equimeasurable with $f$, and for any subcube $R = \prod_{i=1}^n [r_i, r_i + \alpha]$ of $I^n$,

$$\frac{1}{|R|} \int_R |g(x) - f^*(r_i + \alpha)|\,dx_1 \cdots dx_n$$

$$= \frac{1}{\alpha} \int_{r_i}^{r_i + \alpha} [f^*(t) - f^*(r_i + \alpha)]\,dt$$

$$\leq \frac{1}{r_i + \alpha} \int_0^{r_i + \alpha} [f^*(t) - f^*(r_i + \alpha)]\,dt$$

$$= f^*(r_i + \alpha) - f^*(r_i + \alpha) \leq \|f\|_{W(Q)}.$$

Hence $g$ belongs to BMO(Q) and the proof is complete.

The preceding theorem fails when $Q$ is replaced by all of $\mathbb{R}^n$ since BMO($\mathbb{R}^n$) contains functions (such as $\log |x|$) which are unbounded at infinity and hence have decreasing rearrangements which are identically infinite. However, the theorem does contain “local” information pertinent to BMO($\mathbb{R}^n$). For example, when $f$ is in BMO($\mathbb{R}^n$), the inequality (3.3) may be applied to the function $(f - f_Q)\chi_Q$. An integration of both sides produces the basic inequality (4.23) of [1] from which the John-Nirenberg lemma follows easily.

4. Maximal operators

As in the previous section let $Q$ be a fixed cube in $\mathbb{R}^n$. The Hardy-Littlewood maximal function $M_Q f$ of an integrable function $f$ on $Q$ is given by

$$(M_Q f)(x) = \sup_{|Q'| < |Q|} \frac{1}{|Q'|} \int_{Q'} |f(y)|\,dy \quad (x \in Q),$$

where the supremum is taken over all cubes $Q'$ contained in $Q$ and containing $x$. When $Q$ is replaced by all of $\mathbb{R}^n$, the corresponding operator, defined for all locally integrable $f$ on $\mathbb{R}^n$, will be denoted simply by $M$. The next result shows that such maximal operators are bounded on $W$.

**Theorem 4.1.** (a) If $f$ belongs to $W(Q)$, then so does $M_Q f$ and

$$(4.1) \quad \|M_Q f\|_{W(Q)} \leq c \|f\|_{W(Q)},$$

where $c$ depends only on the dimension $n$. 

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(b) The same result holds if \( Q \) is replaced by \( \mathbb{R}^n \) and \( M_q \) by \( M \).

Proof. (a) We may assume that \( f \) is nonnegative. Fix \( t < |Q| \) and let
\[
b = \max(f - f^*(t), 0), \quad g = \min(f, f^*(t)),
\]
so \( f = b + g \). The weak \((1,1)\) and strong \((\infty, \infty)\) properties of \( M_q \) give
\[
(M_q f)^*(t) \leq (M_q b)^*(t-) + (M_q g)^*(0+) \leq ct^{-1}||b||_{L^1} + ||g||_{L^\infty}.
\]
Hence \( (M_q f)^*(t) \) is finite and
\[
0 \leq (M_q f)^*(t) - f^*(t) \leq c \{f^**(t) - f^*(t)\} \quad (t > 0).
\]
Now write
\[
(M_q f)^* - (M_q f)^* = [(M_q f)^* - f**] + [f** - f^*] + [f^* - (M_q f)^*]
\]
and
\[
(M_q f)^*(t) - f^*(t) = \frac{1}{t} \int_0^t [(M_q f)^*(s) - f^*(s)] ds.
\]
Then an application of (4.2) yields
\[
(M_q f)^*(t) - (M_q f)^*(t) \leq c \sup_{0 < s \leq t} \{f^**(s) - f^*(s)\},
\]
from which (4.1) follows. Exactly the same proof establishes part (b).

Next we show that \( M_q \) is a bounded operator on \( \text{BMO}(Q) \). Essentially the same result holds for \( \mathbb{R}^n \) except that functions \( f \) for which \( Mf \) is identically infinite must be ruled out (\( f(x) = \log |x| \) is an example).

**Theorem 4.2.** (a) If \( f \) belongs to \( \text{BMO}(Q) \), then so does \( M_q f \) and
\[
||M_q f||_{\text{BMO}(Q)} \leq c ||f||_{\text{BMO}(Q)},
\]
where \( c \) depends only on the dimension \( n \).

(b) If \( f \) belongs to \( \text{BMO}(\mathbb{R}^n) \), and if \( Mf \) is not identically infinite, then \( Mf \) belongs to \( \text{BMO}(\mathbb{R}^n) \) and
\[
||Mf||_{\text{BMO}(\mathbb{R}^n)} \leq c ||f||_{\text{BMO}(\mathbb{R}^n)}
\]
where \( c \) depends only on \( n \).

Proof. (a) We may assume that \( f \) is nonnegative. Writing \( F \) for the maximal function \( M_q f \) of \( f \), we thus need to show
\[
\frac{1}{|R|} \int_R |F(x) - F_{R'}| dx \leq c ||f||_{\text{BMO}(Q)}
\]
for arbitrary subcubes \( R \) of \( Q \).

Fix \( R \) and let \( 3R \) denote the cube that is concentric with \( R \) and has three times the diameter. Let \( \tilde{R} \) be the smallest subcube of \( Q \) containing
\((3R) \cap Q\), and for each \(x\) in \(R\) let
\[
F_1(x) = \sup \{ f_R : \bar{R} \subset \bar{R} \text{ and } x \in \bar{R} \},
\]
\[
F_2(x) = \sup \{ f_R : \bar{R} \subset Q, x \in \bar{R}, \text{ and } \bar{R} \cap (Q \setminus \bar{R}) \neq \emptyset \}.
\]
Clearly \(F = \max \{ F_1, F_3 \} \) on \(R\) so if \(Q = \{ x \in R : F(x) > F_R \} \), \(Q_1 = \{ x \in Q : F_1(x) > F(x) \}\) and \(Q_2 = Q \setminus Q_1\), then
\[
\int_{Q_1} (F(x) - F_R) dx = \int_{Q_2} (F(x) - F_R) dx.
\]
Hence (4.5) will be established if we show that
\[
\int_{Q_1} (F(x) - F_R) dx = c \frac{1}{|R|} \| f \|_{BMO(Q)}
\]
\((i = 1, 2)\).

Consider first the case \(i = 1\). Since \(f_R \leq F(x) \) for all \(x\) in \(R\), then certainly \(f_R \leq f_R \) so we may construct the Calderón-Zygmund decomposition [9, p. 17] for \(f\) and \(\bar{R}\) with respect to the constant \(F_R\). If the resulting sequence of pairwise disjoint cubes is denoted by \(\{ R_k \}_{k=1}^\infty\), and if \(\bar{R}_k\) denotes the “parent” cube of \(R_k\), then the following properties hold:

(i) \( \bigcup_k R_k \subset \bar{R} \);

(ii) \( f_{\bar{R}_k} \leq F_R < f_{R_k} \) \( (k = 1, 2, \ldots) \);

(iii) \( |\bar{R}_k| = 2^n |R_k| \) \( (k = 1, 2, \ldots) \);

(iv) \( f \leq F_R \) almost everywhere on \( E = \bar{R} \setminus (\bigcup_k R_k) \).

Define functions \(b\) and \(g\) on \(Q\) by
\[
b = \sum_k (f - f_{\bar{R}_k}) 1_{\bar{R}_k}, \quad g = \sum_k f_{\bar{R}_k} 1_{\bar{R}_k} + f 1_{E}
\]
so \(f 1_{\bar{R}} = b + g\). It follows from (ii) and (iv) that
\[
\| g \|_{L^\infty(Q)} \leq F_R,
\]
while on the other hand the John-Nirenberg lemma and (i) and (iii) give
\[
\| b \|_{L^2(Q)} = \left\{ \sum_k \int_{\bar{R}_k} \left| f - f_{\bar{R}_k} \right|^2 dx \right\}^{1/2} \leq \left\{ \sum_k |\bar{R}_k| \int_{\bar{R}_k} \left| f - f_{\bar{R}_k} \right|^2 dx \right\}^{1/2}
\]
\[
\leq c \left( \sum_k 2^n |R_k| \right)^{1/2} \| f \|_{BMO(Q)} \leq c \left| R \right|^{1/2} \| f \|_{BMO(Q)}.
\]
Now it follows from the definition of \(F_1\) that
\[
F_1 \leq M_Q(f 1_{\bar{R}}) = M_Q(b + g) \leq M_Qb + M_Qg,
\]
so applying the Cauchy-Schwarz inequality we obtain
\[
\int_{Q_1} F_1(x) dx \leq \| \Omega_1 \|^{1/2} \| M_Qb \|_{L^2(Q)} + \| \Omega_1 \| \| M_Qg \|_{L^\infty(Q)}
\]
\[
\leq c \left| R \right|^{1/2} \| b \|_{L^2(Q)} + \| \Omega_1 \| \| g \|_{L^\infty(Q)}.
\]
Combining this with (4.7) and (4.8), and subtracting \(\| \Omega_1 \| F_R\) from each side,
we obtain (4.6) for $i = 1$.

The remaining case $i = 2$ will follow directly from the inequality

$$F_2(x) - F_R \leq c ||f||_{BMO(Q)} \quad (x \in \Omega_2)$$

which we now prove. Fix $x$ in $\Omega_2$ and let $P$ be any subcube of $Q$ that contains $x$ and has nonempty intersection with $Q \setminus \bar{R}$.

Clearly $|P| \geq |R|$. Let $P'$ be the smallest subcube of $Q$ containing both $P$ and $R$. Then $|P'| \leq 2^n|P|$. Arguing as before, we note that $f_{P'} \leq F_R$. Hence

$$f_R - F_R \leq f_P - f_{P'} \leq \frac{1}{|P|} \int_P |f(y) - f_{P'}| dy \leq 2^n ||f||_{BMO(Q)},$$

so taking the supremum over all such cubes $P$ we obtain (4.9). This establishes part (a).

The maximal function $F$ in the preceding proof is necessarily integrable over every cube $R$ (contained in $Q$) but this need not be the case when we extend to $\mathbb{R}^n$. However, if $f$ belongs to $BMO(\mathbb{R}^n)$ and $R$ is any cube in $\mathbb{R}^n$, we can split the maximal function $F = Mf$ into the two parts analogous to $F_1$ and $F_2$ in the proof above and estimate these separately. The function $F_1$ is essentially a maximal function relative to a fixed cube and so may be estimated in terms of the $BMO$-norm of $f$ exactly as in the proof above. The function $F_2$ on the other hand is a supremum of averages of $f$ over "large" cubes which, by means of a fixed dilation, may be taken to contain $R$. But then each of these averages is bounded above by the maximal function $Mf$ evaluated at any point of $R$, so $F_2$ is bounded by $\inf_R Mf$. Hence we arrive at the following estimate

$$\frac{1}{|R|} \int_R (Mf)(x) dx \leq c(||f||_{BMO(\mathbb{R}^n)} + \inf_{x \in R} (Mf)(x)).$$

Since $R$ is arbitrary, it follows that the maximal function $F = Mf$ of a function $f$ in $BMO(\mathbb{R}^n)$ is either identically infinite or else it is locally integrable (hence finite a.e. on $\mathbb{R}^n$). In the latter case, having established that the mean $F_R$ is finite, we may proceed exactly as in the proof of part (a) to show that $F$ is in $BMO(\mathbb{R}^n)$. We omit the details.

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References


(Received June 12, 1980)