ON AN INEQUALITY FOR THE SHARP FUNCTION

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The purpose of this note is to refine a rearrangement inequality for \( f^\# \) in terms of the maximal rearrangement \( f^{**} \) for locally integrable functions on \( \mathbb{R}^n \). One consequence of this inequality is an improvement and clarification of the proof that interpolation between \( L^1 \) and \( EO \) "coincides" with that between \( L^1 \) and \( L^\infty \).

The "oscillation" of a locally integrable function \( f \) on \( \mathbb{R}^n \) is gauged by its sharp function (cf [1])

\[
(1) \quad f^\#(x) = \sup_{Q \ni x} \left\{ \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy \right\}, \quad x \in \mathbb{R}^n.
\]

Here \( f_Q \) denotes the average \( \frac{1}{|Q|} \int_Q f(y) \, dy \) of \( f \) over a cube \( Q \) with sides parallel to the coordinate axes, and the supremum in (1) is taken over all such cubes \( Q \) containing the point \( x \). Observe that the sharp function ignores constants: if \( f \) and \( g \) differ by a constant, then \( f^\# \) and \( g^\# \) coincide.

The decreasing rearrangement of \( f \) will be denoted by \( f^{**} \), and its average \( \frac{1}{t} \int_0^t f^{**}(s) \, ds \) by \( f^{**}(t) \). The latter function is "equivalent" to the decreasing rearrangement of the Hardy-Littlewood maximal function \( Mf \) of \( f \) in the sense that the ratio of \( f^{**} \) and \( (Mf)^* \) is contained between positive constants independent of \( f \) (cf [1, Theorem 1.3]).

It follows immediately from (1) that \( f^{**} \leq 2Mf \) and hence, by the preceding remarks, that \( f^{**} \leq cf^{**} \). There is also a result in the opposite direction, that is, it is possible to estimate \( f^{**} \) in terms of \( f^{**} \). In fact, it was shown in [1, Corollary 4.2] that the inequality

\[
(2) \quad f^{**}(t) \leq c \int_t^\infty f^{**}(s) \, ds + f^{**}(\infty), \quad 0 < t < \infty,
\]

holds.
The importance of this result was demonstrated in [1] where a "local" version of (2) easily produced the well-known John-Nirenberg lemma. Nevertheless, the presence of the term $f^{**}(\pm)$ on the right of (2) is bothersome in several of the applications in [1], and this led us to seek a more convenient formulation. In view of a remark made earlier, the first term on the right of (2) is unchanged when a constant $\gamma$ is subtracted from $f$. In this note it will be shown that the finiteness of this integral guarantees the existence of a unique constant $\gamma$ such that $(f-\gamma)^{**}(\pm) = 0$, and hence, if $f$ is replaced by $f-\gamma$ in (2), the troublesome term at infinity does not arise. The main result is thus as follows.

**Theorem 1.** Let $f$ be locally integrable on $\mathbb{R}^n$ and suppose

$$
(3) \quad \int_1^\infty f^{**}(s) \frac{ds}{s} < \infty.
$$

Then there is a unique constant $\gamma$ ($= \lim_{|Q| \to \infty} f_Q$) such that

$$
(4) \quad (f-\gamma)^{**}(t) \leq c \int_t^\infty f^{**}(s) \frac{ds}{s}, \quad 0 < t < \infty.
$$

The proof requires a pair of lemmas.

**Lemma 2.** If $Q_0 \subset Q_1 \subset \mathbb{R}^n$, then

$$
(5) \quad |f_{Q_0} - f_{Q_1}| \leq c \int_{|Q|/2}^\infty f^{**}(s) \frac{ds}{s},
$$

where $c$ depends only on the dimension $n$.

**Proof.** From inequality (4.23) of [1] it follows that

$$
(6) \quad [(f-f_{Q_0})\chi_{Q_0}]^{**}(t) \leq c \int_t^{|Q|} f^{**}(s) \frac{ds}{s}, \quad 0 < t < \frac{|Q|}{2},
$$

for any cube $Q$. But if $0 \leq t \leq |Q|$, then $[\lambda \chi_{Q}]^{**}(t) = \lambda$, so

$$
|f_{Q_0} - f_{Q_1}| = [(f_{Q_0} - f_{Q_1})\chi_{Q_0}]^{**}\left(\frac{|Q|}{2}\right)
$$

$$
= \left[\left(f-f_{Q_0}\right)\chi_{Q_0}\right]^{**}\left(\frac{|Q|}{2}\right) + \left[\left(f-f_{Q_1}\right)\chi_{Q_1}\right]^{**}\left(\frac{|Q|}{2}\right),
$$
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since \((g+h)^{**} \leq g^{**} + h^{**}\) and \(Q_0 \subset Q_1\). Together with (6) (applied for \(Q = Q_0\) and \(Q = Q_1\)) this gives the desired inequality (5).

**Lemma 3.** If condition (3) holds, then \(\lim_{|Q| \to 0} f_Q\) exists.

**Proof.** If \(Q(k)\) denotes the cube with side length \(2^k\) and centered at the origin, then it follows directly from (3) and (5) that \(\{f_Q(k)\}_{k=1}^{\infty}\) is a Cauchy sequence. Let \(\gamma\) be its limit. For each \(c > 0\) it is possible, by (3), to choose a corresponding \(M > 0\) such that

\[
(7) \quad c \int_{M/2}^{\infty} f^{**}(s) \frac{ds}{s} < \frac{\varepsilon}{3},
\]

where \(c\) is the constant in (5). Then \(k\) may be chosen so large that \(|f_Q(k) - \gamma| < \varepsilon/3\) and \(|Q(k)| > M\). In that case, if \(Q\) is any cube with \(|Q| > M\) and if \(Q'\) is any cube containing both \(Q(k)\) and \(Q\), then it follows from (5), (7), and the choice of \(k\) that

\[
|f_Q - \gamma| \leq |f_Q - f_Q'| + |f_Q' - f_Q(k)| + |f_Q(k) - \gamma| < \varepsilon.
\]

**Proof of Theorem 1.** Let \(\gamma = \lim_{|Q| \to 0} f_Q\), which exists by virtue of the preceding lemma. Fix \(t > 0\) and let \(\varepsilon > 0\) be arbitrary. Then there is a cube \(Q\) with measure exceeding \(2t\) and satisfying \(|f_Q - \gamma| < \varepsilon\). Consequently, by (6),

\[
[(f-\gamma)X_Q]^{**}(t) \leq [(f-f_Q)X_Q]^{**}(t) + |f_Q - \gamma|
\]

\[
\leq c \int_{t}^{\infty} f^{**}(s) \frac{ds}{s} < \varepsilon.
\]

If now \(Q \subset \mathbb{R}^n\), the monotone convergence theorem shows that \([(f-\gamma)X_Q]^{**}(t) + (f-\gamma)^{**}(t)\). Hence the preceding estimate, after \(\varepsilon\) is allowed to decrease to zero, produces the desired inequality (4). For the uniqueness, note that if (4) holds for each of two constants \(\gamma_1\) and \(\gamma_2\), then for any \(t > 0\),
\[ |\gamma_1 - \gamma_2| = |\gamma_1 - \gamma_2|^\ast\ast(t) \leq (f - \gamma_1)^{\ast\ast}(t) + (f - \gamma_2)^{\ast\ast}(t) \]
\[ \leq 2c \int_t^\infty f^\#(s) \frac{ds}{s}, \]
and this, by (3), tends to 0 as \( t \to \infty \).

Theorem 1 has a bearing on the identification of the interpolation spaces between the Lebesgue space \( L^1 \) and the space \( \text{BMO} \) of functions of bounded mean oscillation. The latter space consists of all equivalence classes (denoted by \( F \)) modulo constants of functions \( f \) for which \( f^\# \) is bounded. Since the sharp function is invariant under addition of constants the notation \( F^\# = f^\# \) for any representative \( f \) of \( F \) is meaningful. Hence, with the norm

\[ \|F\|_{\text{BMO}} = \|F^\#\|_{L^\infty}, \]

\( \text{BMO} \) is a Banach space.

It was established in [1,§6] that the Peetre \( K \)-functional

\[ K(f,t;L^1,\text{BMO}) = \inf_{g = h + t} \|g\|_{L^1} + t\|H\|_{\text{BMO}} \]

is given by

\[ K(f,t;L^1,\text{BMO}) \sim tf^\#(t), \quad 0 < t < \infty, \]

for any \( f \) in \( L^1 + \text{BMO} \).

Notice, however, that \( (L^1,\text{BMO}) \) is not, strictly speaking, a compatible Banach couple in that one space consists of functions and the other of equivalence classes of such functions modulo constants. The difficulty is resolved by introducing the space \( L^1 \) consisting of all equivalence classes \( F \) for which the norm \( \|F\|_{L^1} = \inf_{f \in F} \|f\|_{L^1} \) is finite. Of course there will be precisely one representative \( f \) in \( L^1 \) of each equivalence class \( F \) in \( L^1 \). With this, it is clear from (9) that the analogous result

\[ K(F,t;L^1,\text{BMO}) \sim tf^\#(t) \]
holds for the Banach couple \((l^1, \text{BMO})\). Hence, with the aid of Theorem 1, the \((\theta, q)\)-interpolation spaces may be identified as follows.

**Corollary 4.** Suppose \(0 < \theta < 1\), \(0 < q \leq \infty\), and let \(l/p = 1 - \theta\). Then, for any \(F\) in \(l^1 + \text{BMO}\),

\[
\|F\|_{(l^1, \text{BMO})_{\theta, q}} \sim \|F\|_{L^p q} = \|f\|_{L^p_q} \sim \|f - \gamma\|_{L^p_q},
\]

where \(f\) is any representative of \(F\) and \(\gamma = \lim_{|Q| \to +} f_Q\). Hence

\[
(l^1, \text{BMO})_{\theta, q} = (L^{1, \text{BMO}})_{\theta, q} = L^p_q,
\]

in the sense that if \(F\) belongs to \((l^1, \text{BMO})_{\theta, q}\), then there is a unique representative \(g = f - \gamma\) of \(F\) such that

\[
\|g\|_{L^p_q} \leq c \|F\|_{(l^1, \text{BMO})_{\theta, q}},
\]

and, conversely, that if \(f\) belongs to \(L^p_q\), then

\[
\|f\|_{(l^1, \text{BMO})_{\theta, q}} \leq c \|f\|_{L^p_q},
\]

where \(F\) is the coset containing \(f\).

**Remarks.** (i) The identification of the interpolation spaces, as in Corollary 4, does not require the complete knowledge of (9) of the \(K\)-functional (cf [1, Corollary 4.4]).

(ii) The Hilbert transform and its \(n\)-dimensional analogues (the Riesz transforms) are well defined on \(L^p\) \((1 \leq p \leq \infty)\) as the principal value integrals

\[
(11) \quad R_f(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} f(y)K_\varepsilon(x-y)dy
\]

where \(K_\varepsilon(y) = K(y)\chi_{\{\varepsilon, \infty\}}(|y|)\) and \(K(y) = c_n y_\parallel y^{n+1}\). On \(L^\infty\), however, these integrals will normally not exist and so must be modified [2] according to...
\[(12) \quad \tilde{R}_j f(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} f(y) [K_\varepsilon(x-y) - K_1(y)] dy.\]

To use the facts that the "Hilbert transform" maps $L^\infty$ into BMO and say, $H^1$ into $L^1$ (or, $L^1$ into weak $L^1$) together with interpolation in order to obtain intermediate results, the ambiguity between the definitions (11) and (12) must be resolved. The inequality one is able to obtain [1, Corollary 5.6] is

\[(13) \quad (R_j f)^*(t) \leq c \left\{ \frac{1}{t} \int_0^t f^*(s) ds + \int_t^\infty f^*(s) \frac{ds}{s} \right\}\]

where the left hand side makes sense whenever the right hand side of the inequality is finite. This inequality is derived by using the finiteness of $\int_t^\infty f^*(s) \frac{ds}{s}$ to establish that

\[\int_{\mathbb{R}^n} |f(y)| \left| K_1(y) \right| dy < \infty\]

and so (11) and (12) differ by at most a constant $\gamma$ which is controlled according to Theorem 1.

REFERENCES


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