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WEAK-TYPE INEQUALITIES FOR H^p AND BMO

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ABSTRACT. In connection with the Marcinkiewicz interpolation theorem, it appears that the most convenient way of storing the weak-type information for a given operator is in terms of a single inequality called a weak-type inequality. The first part of the paper surveys recent results on weak-type inequalities, together with their applications in harmonic analysis and approximation theory. The second part contains some new results on weak-type inequalities in the H^p-theory. These include a characterization of the Peetre K-functional for (L^p, BMO) in terms of the sharp-function, and a weak-type inequality for the sharp-function which leads to a simple proof of the John-Nirenberg lemma for functions of bounded mean oscillation.

§1. THE HARDY-LITTLEWOOD MAXIMAL OPERATOR AND REARRANGEMENTS. The Hardy-Littlewood maximal function of a locally integrable function f on R^n is given by

$$(1.1) \qquad (Mf)(x) = \sup_{Q \ni x} \{\frac{1}{|Q|} \int_Q |f(y)|dy\},$$

where the supremum extends over all cubes Q containing x with sides parallel to the coordinate axes. From its origins in function theory [13], it has evolved into an important tool in harmonic analysis [27] and related areas such as probability [23] and ergodic theory [11].

The maximal function takes into account the local, as opposed to the pointwise, behavior of f. It thus provides a representation of the "magnitude" of f amenable to differentiation and integration theory. Quantitative measurement of the magnitude is most naturally made by expressing the function as a member of such function spaces as L^p, L^{pq}, $L^p (\log L)^\alpha$, etc. Hence, most

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applications hinge on the boundedness of the maximal operator M between suitably chosen pairs of rearrangement-invariant spaces. The purpose of this section is to demonstrate that not only are the norms of Mf and f related in various ways but that in fact all such estimates are consequences of a simple relationship between the decreasing rearrangements of the functions themselves (cf. Theorem 1.3).

Indeed, the function

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \qquad t > 0, \tag{1.2}$$

clearly resembles the maximal function of the decreasing rearrangement f^* of f. Theorem 1.3, to the effect that $(Mf)^* \sim f^{**}$, tells us that f^{**} is also the decreasing rearrangement of the maximal function Mf of f. In particular, for rearrangement-invariant estimates, the maximal function Mf can always be replaced by the simpler function f^{**}, which is itself nothing more than an average of f^*.

The idea that $(Mf)^*$ is dominated by f^{**} goes back in some form to Hardy-Littlewood [13]; cf. also [29, pp. 29-33]. The equivalence of these two functions seems however to have first been pointed out by Herz [14]. In this section we shall present a simple proof of Herz' theorem (Theorem 1.3). The following covering lemma will be needed.

LEMMA 1.1. Let Ω be an open subset of R^n with finite measure. Then there are dyadic cubes $Q_j, j = 1,2,3,\ldots$, with pairwise disjoint interiors, such that

a) $$Q_j \cap \Omega^c \neq \phi, \qquad \text{for each } j;$$

b) $$\Omega \subseteq \bigcup_j Q_j;$$

c) $$|\Omega| \le \sum_j |Q_j| \le 2^n|\Omega|.$$

Proof. For each $x\in\Omega$, select a dyadic cube, $Q(x)$ say, of smallest diameter, which contains x and has nonempty intersection with Ω^c. Now subdivide $Q(x)$ into 2^n congruent subcubes and select any one, $\tilde{Q}(x)$ say, which contains x. Clearly $\tilde{Q}(x) \subset \Omega$, and so

$$2^{-n}\,|Q(x)| = |\tilde{Q}(x)| = |\tilde{Q}(x) \cap \Omega| \le |Q(x) \cap \Omega|. \tag{1.3}$$

Let $K = \{Q(x) : x\in\Omega\}$. Because of the dyadic nature of the cubes and the fact that $|\Omega| < \infty$, each $x\in\Omega$ is contained in a maximal cube, say $\overline{Q}(x)$, from K. Listing the at most countably many cubes in $\{\overline{Q}(x) : x\in\Omega\}$ as $Q_1, Q_2, \ldots$, we see that properties a) and b) are immediate, hence so is the first inequality in c). The remaining inequality follows from the observation that (1.3) is valid for every $Q_j : 2^{-n}|Q_j| \le |Q_j \cap \Omega|$; summing over j, we obtain the desired result.

REMARK 1.2. If Q is a fixed cube containing Ω, and if Ω is open relative to Q, then a similar argument shows that the cubes Q_j can be selected as subcubes of Q (i.e., dyadic with respect to Q) and each Q_j meets $Q\backslash\Omega$.

THEOREM 1.3 (Herz [14]). If f is locally integrable on R^n, then

$$3^{-n}(Mf)^*(t) \le f^{**}(t) \le (2^n+1)(Mf)^*(t), \qquad t > 0. \tag{1.4}$$

Proof. Fix $t > 0$. For the right-hand inequality we can suppose $(Mf)^*(t) < \infty$. The lower semi-continuity of Mf guarantees that the set $\Omega = \{x\varepsilon R^n : (Mf)(x) > (Mf)^*(t)\}$ is open, and the estimate $|\Omega| \le t$ follows from the equimeasurability of Mf and $(Mf)^*$. Applying Lemma 1.1 to Ω, we obtain a sequence of cubes Q_j, with disjoint interiors, for which properties a), b), and c) of the lemma hold. With $F = (\cup Q_j)^c$, we put $g = \sum_j f\chi_{Q_j}$ and $h = f\chi_F$. The subadditivity of $f \to f^{**}$ gives immediately

$$f^{**}(t) \le g^{**}(t) + h^{**}(t) \le t^{-1}\|g\|_1 + \|h\|_\infty. \tag{1.5}$$

But $F \subset \Omega^c$ so

$$\|h\|_\infty \le \|\chi_F Mf\|_\infty \le (Mf)^*(t). \tag{1.6}$$

Furthermore, each Q_j meets Ω^c so $|Q_j|^{-1}\int_{Q_j} |f(x)|dx \le (Mf)^*(t)$. Hence by Lemma 1.1 c),

$$\|g\|_1 = \sum_j \int_{Q_j} |f(x)|dx \le 2^n|\Omega|(Mf)^*(t) \le 2^n t(Mf)^*(t).$$

Together with (1.5) and (1.6), this gives the desired result.

In the following argument to establish the left-hand inequality in (1.4) we use the fact that the maximal operator is of weak type (1,1) ($t(Mf)^*(t) \le 3^n\|f\|_1$) and of strong type (∞,∞) ($\|Mf\|_\infty \le \|f\|_\infty$), and repeat a standard argument due originally to Calderón [5, Theorem 8]. Thus, we assume $f^{**}(t) < \infty$ and consider the set $E = \{x\varepsilon R^n : |f(x)| > f^*(t)\}$. Again we have $|E| \le t$. For the functions

$$g(x) = (f(x) - f^*(t)\operatorname{sgn} f(x))\chi_E(x), \qquad h = f-g,$$

we have the estimates

$$\|g\|_1 \le t(f^{**}(t) - f^*(t)), \qquad \|h\|_\infty \le f^*(t).$$

Since

$$(Mf)(x) \le (Mg)(x) + (Mh)(x) \le (Mg)(x) + \|h\|_\infty,$$

it follows that

$$(Mf)^*(t) \le (Mg)^*(t) + \|h\|_\infty \le 3^n t^{-1}\|g\|_1 + \|h\|_\infty$$
$$\le 3^n(f^{**}(t) - f^*(t)) + f^*(t) \le 3^n f^{**}(t).$$

This completes the proof.

Herz' theorem, which asserts that $(Mf)^* \sim f^{**}$, enables us to use f^{**} as a "model" for Mf. The Hardy-Littlewood maximal theorem [29, p.32] is an easy consequence since $\|f^{**}\|_p = \|\frac{1}{t}\int_0^t f^*(s)ds\|_p \le c_p\|f^*\|_p$ $(p > 1)$, by virtue of a classical inequality of Hardy [28, p.196].

COROLLARY 1.4 (Hardy-Littlewood). Suppose $1 < p \le \infty$. If $f \in L^p(R^n)$, then $Mf \in L^p(R^n)$ and $\|Mf\|_p \le c_p\|f\|_p$.

The model reveals much more. An interchange in the order of integration gives

$$(1.7) \qquad \int_0^1 f^{**}(t)dt = \int_0^1 \frac{1}{t}\int_0^t f^*(s)ds\,dt = \int_0^1 f^*(s)ds\int_s^1 \frac{dt}{t} = \int_0^1 f^*(s)\log\frac{1}{s}\,ds.$$

If we work on the unit circle T, say, (or any fixed ball in R^n) so that the decreasing rearrangements vanish outside of a finite interval (0, 1), then the right-hand side is a norm for the space $L\log^+L$ (T) [2]. The left-hand side is, by Herz' theorem, equivalent to the L^1-norm of Mf. Thus, on the one hand, we obtain the Hardy-Littlewood result [29, p.32] that $f \in L\log^+L$ implies $Mf \in L^1$, and on the other, the Stein [26]-Herz [14] converse : $Mf \in L^1$ implies $f \in L\log^+L$.

COROLLARY 1.5 (Hardy-Littlewood-Stein-Herz). Suppose $f \in L^1(T)$. Then Mf is integrable if and only if $f \in L\log^+L(T)$.

It should come as no surprise that Corollaries 1.4 and 1.5 fall out so easily when we remark that the Marcinkiewicz interpolation theorem [30, p.112] is lurking in the background. Indeed, we used the usual weak-type hypotheses for M in the proof of Herz' theorem. The fundamental relationship (1.4) is seen therefore as a convenient and concise way of storing the weak-type information for the maximal operator (strictly speaking, this remark applies only to the left-hand inequality in (1.4); the right-hand inequality is a bonus: it gives us the Stein-Herz converse).

In the next section we shall see how this program can be repeated for other basic weak-type operators such as the Hilbert transform and the fractional integrals. This leads to a new way of viewing the Marcinkiewicz interpolation theorem, and to significant extensions (cf. Section 3) of that theorem beyond its traditional domain in harmonic analysis.

§2. WEAK-TYPE INEQUALITIES. The conjugate function, or periodic Hilbert transform, $Hf = \tilde{f}$ of a function $f \in L^1(T)$ is defined [29, p.131] by the principal-value integral

$$(Hf)(e^{it}) \equiv \tilde{f}(e^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{ix}) \cot\left(\frac{t-x}{2}\right)dx.$$

Our point of departure is the O'Neil-Weiss [21] inequality

$$(2.1) \qquad (Hf)^{**}(t) \le c\left(\frac{1}{t}\int_0^t f^{**}(s)ds + \int_t^1 f^{**}(s)\,\frac{ds}{s}\right), \qquad 0 < t < 1,$$

which is a fairly easy consequence of the Stein-Weiss [28, p.240] description of $(H\chi)^*$ for characteristic functions χ. Clearly, the inequality has meaning only if f^{**} is integrable, that is, if $f \in L \log L$ (cf. (1.7)). Thus, (2.1) does not describe the action of H on all of L^1 and, in particular, does not explicitly contain the information that H is of weak type (1,1) [29, p.134]. Nevertheless, this additional information can be incorporated into (2.1) by means of an elementary decomposition argument similar to that used in the second half of the proof of Theorem 1.3. In so doing, Bennett-Rudnick [2] established the following inequality

$$(2.2) \qquad (Hf)^{*}(t) \le c\left(\frac{1}{t}\int_0^t f^{*}(s)ds + \int_t^1 f^{*}(s)\frac{ds}{s}\right), \qquad 0 < t < 1.$$

valid now for all $f \in L^1$. Note that (2.1) follows, by integration, from (2.2).

This is the exact counterpart, for the Hilbert transform, of the fundamental inequality (1.4) for the maximal operator. As before, the classical estimates are easy consequences:

<u>COROLLARY 2.1</u> a) (M. Riesz [29, p.253]) $H : L^p \to L^p$, $1 < p < \infty$;
b) (Zygmund [29, p.254]) $H : L \log L \to L^1$;
c) (Zygmund [29, p.254]) $H : L^\infty \to L_{exp}$.

Proof. a) The averaging operator $t^{-1}\int_0^t (.)\,ds$ is bounded on L^p for $p > 1$ (by Hardy's inequality); its adjoint $\int_t^1 (.)ds/s$ is bounded on L^p for $p < \infty$.
b) Integrate each side of (2.1) and use (1.7).
c) If f is bounded, then (2.2) shows that $(Hf)^*(t)$ grows at most logarithmically as $t \to 0$.

What is the interpretation in terms of the Marcinkiewicz interpolation theorem? Certainly H is of weak type (1,1) but on the other hand there seems to be no reasonable way of defining a concept of weak type (∞, ∞) that will be satisfied by the Hilbert transform. And yet precisely this kind of information seems to be encoded in the inequality (2.2) because of the results it produces in Corollary 2.1.

We can better understand what is happening here by considering the Weyl fractional integrals I_λ, $0 < \lambda < 1$ [30, p.135]:

$$(I_\lambda f)(e^{it}) = \frac{1}{\Gamma(\lambda)} \int_0^\infty f(e^{i(t-x)})x^{\lambda-1}dx \quad ^{1)}$$

The operator I_λ is of weak types $(1, (1-\lambda)^{-1})$ and (λ^{-1}, ∞) (cf. [2]). Now we invoke the fundamental contribution of Calderón [5, p.290]: so long as p_o and p_1 are finite, the pair of weak-type conditions (p_o,q_o) and (p_1,q_1) on an operator T can always be combined and, in fact, are equivalent to a single inequality satisfied by T. In the case of the operator I_λ, the inequality is

(2.3) $$(I_\lambda f)^*(t) \le c\,(t^{\lambda-1}\int_0^t f^*(s)ds + \int_t^1 s^\lambda f^*(s)ds/s), \qquad 0 < t < 1.$$

The point of the exercise is that the Hilbert transform is, in a formal sense, the fractional integral I_o of order 0. Letting $\lambda \to 0$, we see that the weak-type condition $(1, (1-\lambda)^{-1})$ tends to weak type (1,1), but the condition (λ^{-1},∞) tends to the meaningless weak type (∞,∞). However, the equivalent inequality (2.3) tends to precisely the inequality (2.2)! This suggests that we adopt the Calderón formulation, extended to infinite values of the parameters, as the fundamental notion of weak type.

DEFINITION 2.2 [2]. Suppose $0 < p_o < p_1 \le \infty$, $0 < q_o, q_1 \le \infty$, with $q_o \ne q_1$, and let m be the slope of the line segment σ joining the points $(1/p_o, 1/q_o)$ and $(1/p_1, 1/q_1)$ in the plane : $m = (1/q_1 - 1/q_o)/(1/p_1 - 1/p_o)$. Let $S(\sigma)$ be the integral operator defined by

(2.4) $$S(\sigma)f(t) = t^{-1/q_o}\int_0^{t^m} s^{1/p_o} f(s)\frac{ds}{s} + t^{-1/q_1}\int_{t^m}^\infty s^{1/p_1} f(s)\frac{ds}{s}, \quad 0 < t < \infty .$$

We say that an operator T is of weak type $(p_o, q_o; p_1, q_1)$ if

(2.5) $$(Tf)^*(t) \le cS(\sigma)(f^*)(t), \qquad 0 < t < \infty ,$$

for all f for which the right-hand side is finite.

In particular, (2.2) shows that the Hilbert transform is of weak type $(1,1;\infty,\infty)$, and (2.3) that the fractional integrals I_λ are of weak type $(1, 1/(1-\lambda); 1/\lambda,\infty)$. Any inequality of the form (2.5) will be referred to simply as a weak-type inequality.

Once the weak-type inequality (2.5) is established for a given operator T, the interpolation is performed exactly as in Corollary 2.1(a) by means of the Hardy inequalities. Thus, if $0 < \theta < 1$ and

(2.6) $$\frac{1}{p} = \frac{1-\theta}{p_o} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_o} + \frac{\theta}{q_1},$$

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1) defined in this way for f with mean value 0 on T, and extended by linearity to all $f \in L^1$.

we apply the Lorentz L^{qr}-norm to each side of (2.5). The operator $S(\sigma)$ is so designed that the right-hand side reduces, via the Hardy inequalities, to the L^{pr}-norm of f. This shows that $T : L^{pr} \to L^{qr}$, for any r, which is precisely Calderón's formulation [5, p.293] (and proof) of the Marcinkiewicz interpolation theorem.

When applied to the Hilbert transform, this result produces the M. Riesz theorem as presented in Corollary 2.1(a). It is natural to ask whether parts (b) and (c), which involve the Zygmund spaces L log L and L exp, can also be derived in this way. While such spaces have traditionally been regarded as Orlicz spaces (thus preventing their incorporation in the Calderón theory), it is nevertheless the case that they can also be regarded as more general types of Lorentz spaces, and can therefore be easily amalgamated with the L^{pq}-spaces. The appropriate framework is furnished by the class of Lorentz-Zygmund spaces $L^{pq}(\log L)^{\alpha}$, introduced by Bennett-Rudnick [2]. A function f (on the circle, say) is in $L^{pq}(\log L)^{\alpha}$, $0 < p,\ q \le \infty$, $-\infty < \alpha < \infty$, if

$$(2.7)\qquad \|f\|_{p,a;\alpha} = \left(\int_0^1 [t^{1/p}(1-\log t)^{\alpha} f^*(t)]^q \frac{dt}{t}\right)^{1/q} < \infty$$

(with the evident modification if $q = \infty$).

Clearly, $L^{pq}(\log L)^{0}$ is the familiar Lorentz space L^{pq}, and it is not hard to show that $L^{pp}(\log L)^{\alpha}$ is the Zygmund space $L^{p}(\log L)^{\alpha}$ when $p < \infty$. The Zygmund space of α-th power exponentially-integrable functions is nothing more than the space $L^{\infty\infty}(\log L)^{-1/\alpha}$. Furthermore, the O'Neil spaces $K^{p}(\log^{+}K)^{\alpha p}$ [18] arise as the Lorentz-Zygmund spaces $L^{p1}(\log L)^{\alpha}$. Complete details are given in [2].

With these foundations in place, it remains to formulate the Marcinkiewicz interpolation theorem in terms of operators of weak type $(p_o,q_o;p_1,q_1)$ acting on Lorentz-Zygmund spaces. The first part gives the "internal" results corresponding to the values $0 < \theta < 1$ in (2.6); the other two parts give the "end-point" results corresponding to $\theta = 0$ and $\theta = 1$.

<u>THEOREM 2.3</u> [1] (Bennett-Rudnick [2]). Suppose $0 < p_o < p_1 \le \infty$ and $0 < q_o,\ q_1 \le \infty$, with $q_o \ne q_1$. Let T be a quasilinear operator of weak type $(p_o,q_o;p_1,q_1)$.

a) If $0 < \theta < 1$ and p, q are given by (2.6), then

$$T : L^{pa}(\log L)^{\alpha} \to L^{qa}(\log L)^{\alpha}$$

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[1] We present only the finite measure space version; the general case is given in [2].

whenever $0 < a \le \infty$ and $-\infty < \alpha < \infty$.

b) If $1 \le a \le b \le \infty$ and $-\infty < \alpha, \beta < \infty$, then

$$T : L^{p_0 a}(\log L)^{\alpha+1} \to L^{q_0 b}(\log L)^{\beta}$$

whenever $\alpha + 1/a = \beta + 1/b > 0$.

c) If $1 \le a \le b \le \infty$ and $-\infty < \alpha, \beta < \infty$, then

$$T : L^{p_1 a}(\log L)^{\alpha+1} \to L^{q_1 b}(\log L)^{\beta}$$

whenever $\alpha + 1/a = \beta + 1/b < 0$.

The essence of the result is that the "index" $\sigma + 1/s$ of the space $L^{rs}(\log L)^{\sigma}$ remains constant when $0 < \theta < 1$ but always decreases by a factor of one in the endpoint cases $\theta = 0$ and $\theta = 1$. This single result directly produces all of the classical rearrangement-invariant estimates for such fundamental operators as the Hilbert transform, the fractional integrals, the maximal operator, and the Fourier transform; complete details can be found in [2]. Furthermore, as DeVore-Riemenschneider-Sharpley [7] have shown, this natural formulation of the Marcinkiewicz theorem lifts effortlessly into a general Banach space context and hence produces further applications in harmonic analysis and approximation theory. These results form the core of the next section.

We conclude our discussion of the rearrangement-invariant case with some remarks on multilinear generalizations of Theorem 2.3. Such results are of importance in dealing with convolution and tensor product operators [19,20,25], for example. Sharpley [25] has developed weak-type inequalities for bilinear (or multilinear) operators T satisfying m individual weak-type estimates. The weak-type inequality has the form

$$T(f,g)^{**}(t) \le c \int_0^\infty \int_0^\infty f^*(r)g^*(s)\, \Psi_\sigma(r,s;t)\, \frac{dr}{r}\frac{ds}{s} \equiv S_\sigma(f^*,g^*)(t) \tag{2.8}$$

where the kernel Ψ_σ is a combination of powers of r, s, and t determined by the m initial estimates.

The "internal" mapping properties of T are obtained from (2.8) exactly as in the linear case (Theorem 2.3a)), namely, by applying appropriate norms to (2.8) and reducing the right-hand side by means of suitable generalizations of the Hardy inequalities (cf. [25]). The analysis of the endpoint cases in Theorem 2.3 b), c) is much more intricate. Nevertheless, it again ultimately depends on what can be regarded as limiting cases of the Hardy inequalities. It would be of some interest to have corresponding inequalities for the multilinear theory.

§3. GENERALIZED WEAK-TYPE INEQUALITIES. The results of the first two sections have been concerned with rearrangements and hence with the magnitude of the function. We now want to consider other characteristics of the function such as its smoothness. DeVore-Riemenschneider-Sharpley [7] made the interesting observation that while magnitude and smoothness are unrelated, the analysis in the two situations is exactly the same. Indeed, for the periodic Hilbert transform H there is the inequality [29, p.121]

$$(3.1) \qquad \frac{\omega(Hf;t)}{t} \le c\left(\frac{1}{t}\int_0^t \frac{\omega(f;s)}{s}\,ds + \int_t^1 \frac{\omega(f;s)}{s}\frac{ds}{s}\right), \qquad 0 < t < 1,$$

where $\omega(g;.)$ is the modulus of continuity [29, p.42] of a function g. This inequality has exactly the same structure as (2.1)(or (2.2)) but with f^{**} (or f^*), the measure of magnitude, replaced by $t^{-1}\omega(f;t)$, the measure of smoothness. Proceeding as in Section 2, we obtain the precise analogues of the results in Corollary 2.1. Thus $H : \mathrm{Lip}(\alpha,q) \to \mathrm{Lip}(\alpha,q)$ if $0 < \alpha < 1$; at the endpoints we find that $H : D \to C$, where D is the Dini class and C the space of continuous functions; and if $f \in \mathrm{Lip}\ 1$, then $\omega(Hf;t)/t$ grows at most logarithmically as $t \to 0$. In fact, by applying the Lorentz-Zygmund norms to (3.1), we obtain the whole spectrum of results corresponding to Theorem 2.3.

The crucial link between (2.1) and (3.1), which allows the abstract theory to unfold, is provided by the Peetre K-functional [4, Chapter 3]. If $f \in X_o + X_1$, where (X_o,X_1) is a compatible couple of Banach spaces, then the K-functional $K(f;t) \equiv K(f;t;X_o,X_1)$ is defined by

$$(3.2) \qquad K(f;t) = \inf_{f=f_o+f_1} \left(\|f_o\|_{X_o} + t\|f_1\|_{X_1}\right), \qquad 0 < t < \infty,$$

where the infimum is taken over all possible representations $f = f_o + f_1$ of f with $f_o \in X_o$ and $f_1 \in X_1$.

The point is that $t^{-1}K(f;t;L^1,L^\infty) = f^{**}(t)$ and $t^{-1}K(f;t;C,C^{(1)}) \sim t^{-1}\omega(f;t)$ [4, Chapter 3]. Hence, both (2.1) and (3.1) involve particular kinds of K-functionals related by means of a weak-type inequality.

DEFINITION 3.1 [7]. Let (X_o,X_1), (Y_o,Y_1) be compatible couples of Banach spaces. Suppose $1 \le p_o \le p_1 \le \infty$, $1 \le q_o, q_1 \le \infty$, $q_o \ne q_1$, and let σ be the corresponding interpolation segment (cf. Definition 2.2). Let T be a quasi-linear operator carrying $X_o + X_1$ into $Y_o + Y_1$. Then T is said to be of (generalized) weak-type σ with respect to (X_o,X_1) and (Y_o,Y_1) if

$$(3.3) \qquad \frac{K(Tf;t;Y_o,Y_1)}{t} \le cS(\sigma)\left[\frac{K(f;(.);X_o,X_1)}{(.)}\right](t), \qquad 0 < t < \infty,$$

holds whenever the right-hand side is finite.

Once such an inequality has been established, the mapping properties of T are obtained as before by applying Lorentz-Zygmund norms and obtaining results analogous to Theorem 2.3. We shall not discuss the mapping properties in any great detail here. Instead, we shall try to convey some of the flavor of the theory by pointing out some of the interesting weak-type inequalities and the phenomena they control. For further details, see DeVore-Riemenschneider-Sharpley [7], or the survey article [3].

The smoothness spaces derived from L^p, namely the Besov spaces $B_p^{\lambda,a}$, are defined in terms of the k-th order L^p-modulus of continuity

$$\omega_k(f;t)_p = \sup_{|h|\le t} \|\Delta_h^k f\|_p, \tag{3.4}$$

where Δ_h^k is the k-th power of the difference operator $(\Delta_h f)(x) = f(x+h) - f(x)$. It is a well-known result that $B_p^{\lambda,a}$ is independent of the order k used in its definition, provided only that $k > \lambda$ [4, Chapter 3]. Lying beneath is a weak-type inequality known as Marchaud's inequality [16]. We consider only the simplest case, involving $0 < \lambda \le 1$ and $k = 1$ or 2. The trivial "direct" estimate

$$\omega_2(f;t)_p \le 2\omega_1(f;t)_p \tag{3.5}$$

follows directly from the definition (3.4). The corresponding "inverse" result is given by Marchaud's inequality:

$$t^{-1}\omega_1(f;t)_p \le c\{\|f\|_p + \int_t^\infty s^{-1}\omega_2(f;s)_p\, ds/s\}. \tag{3.6}$$

Now the Besov space $B_p^{\lambda,a}$ results from applying the Lorentz $L^{1/(1-\lambda),a}$ norm to $t^{-1}\omega_k(f;t)$. Clearly, in view of (3.5) and (3.6), it is immaterial which of $k = 1$ or $k = 2$ is used, provided only that the averaging operator $\int_t^\infty (\cdot)ds/s$ remains bounded; as we have noted previously, this is the case when $1/(1-\lambda) < \infty$, that is, when $\lambda < 1$. When $\lambda = 1$, the averaging operator is unbounded and in fact it is well-known that in this case the spaces corresponding to $k = 1$ and $k = 2$ no longer coincide (cf. [29, p.47]).

Note that since the K-functionals on the left- and right-hand sides of (3.3) are not necessarily the same, the inequality can store important information even for the identity operator: Marchaud's inequality is an example. Another example arises in the Sobolev-type embedding theorems for Besov spaces. Here we want to compare two k-th order L^p-moduli but this time with k fixed and p varying. The inequality reads

$$\omega_k(f;t)_q \le c\int_0^t s^{-\theta}\omega_r(f;s)_p ds/s, \qquad 0 < t < \infty, \tag{3.7}$$

where $\theta = n/p - n/q$ (n the euclidean dimension) and $q > p$ [7]. The usual embeddings of $B_p^{\lambda,a}$ into $B_q^{\mu,b}$, together with the logarithmic end-point estimates, all follow directly by applying Lorentz-Zygmund norms to (3.7). A variant of

(3.7), in which the modulus on the left-hand side is replaced by f^{**}, produces embeddings of Besov spaces into rearrangement-invariant spaces such as L^q, L^{qa}, etc. [7].

The Bernstein-type theorems on the absolute convergence of Fourier series [29, p.243] are also controlled by weak-type inequalities. Such theorems relate the magnitude of the Fourier coefficients $\hat{f}(n)$ to the smoothness of f. The relevant inequality is [7]

$$(\hat{f})^{**}(t) \le c\frac{1}{t}(\|f\|_1 + \int_{1/t}^{1} \frac{\omega(f;s)_p}{s^{1/p}}\frac{ds}{s}), \qquad 1 \le t \le \infty,$$

valid for $f \in L^p$, $1 \le p \le 2$.

The weak-type inequalities also play an important role in approximation theory. For example, when one of the K-functionals in (3.3) is chosen to measure smoothness, the other can be replaced by the closely related "degree of approximation" functional E(f;t). Thus, in $L^p(T)$ for instance,

$$E(f;t)_p = \inf\{\|f-g\|_p\}$$

where the infimum is taken over all trigonometric polynomials g whose degree does not exceed [t]. The corresponding weak-type inequalities give rise to some of the classical "direct" and "inverse" approximation theorems; see [7] for details.

The moral of the last three sections, if it is not already clear, is that of the numerous phenomena in analysis that can be interpreted in terms of an operator acting between a pair of spaces, a great many are controlled by a weak-type inequality relating an appropriate pair of K-functionals. Furthermore, when it is known that the operator is unbounded at one or other of the "endpoints" (and this, after all, is what the Marcinkiewicz theorem is all about), then it is often possible to anticipate, by including one or the other of the averaging operators $t^{-1}\int_0^t(.)ds$, $\int_t^\infty(.)ds/s$, exactly what form the weak-type inequality must take.

We shall see some good examples of this in the next section when we turn to the theory of H^p-spaces and BMO. The basic characteristic of H^1-functions is not smoothness or magnitude, but something in between (one hesitates to call it analyticity!). Whatever one calls it, it is expressed in terms of the grand maximal function $\mathcal{M}f$, and this (more precisely, $(\mathcal{M}f)^{**}(t)$) is equivalent to $t^{-1}K(f;t;H^1,L^\infty)$ [9]. Similarly, the basic characteristic of BMO-functions is oscillation which is expressed in terms of the sharp-function $f^{\#}$. We shall show (in Section 6) that $f^{\#*}(t)$ is equivalent to $t^{-1}K(f;t;L^1,\mathrm{BMO})$. These key descriptions of the K-functionals lead to some interesting weak-type inequalities which play a fundamental role in the theory. For example, the weak-type inequality (4.15) relating magnitude (f^{**}) and oscillation ($f^{\#*}$) will lead us to a simple proof of the John-Nirenberg lemma and other results.

§4. INTERPOLATION BETWEEN L^1 AND BMO. The space BMO of functions of bounded mean oscillation was devised by John-Nirenberg [15] for the purpose of studying regularity properties of solutions of elliptic partial differential equations. BMO is the Banach space (of equivalence classes modulo the constants) of all locally integrable functions f on R^n for which

$$\|f\|_{BMO} = \sup_{Q} Q(|f - Q(f)|)$$

is finite, where $Q(f) = |Q|^{-1}\int_Q f(y)dy$ and the supremum is taken over all cubes Q with sides parallel to the coordinate axes.

The BMO-norm measures the oscillation of f on all of R^n. The local oscillation is expressed by the "sharp-function"

$$f^{\#}(x) = \sup_{Q\ni x} Q(|f - Q(f)|) \tag{4.1}$$

which was introduced by Fefferman-Stein [10]; the supremum is now taken over only those cubes Q which contain the point x. Clearly, $f \in BMO$ if and only if $f^{\#} \in L^{\infty}$, and

$$\|f\|_{BMO} = \|f^{\#}\|_{\infty}. \tag{4.2}$$

The space BMO could just as well be defined in terms of the quantity

$$f^{b}(x) = \sup_{Q\ni x}\{\inf_{c} Q(|f-c|)\}; \tag{4.3}$$

indeed, it is easily verified that

$$f^{b}(x) \le f^{\#}(x) \le 2f^{b}(x). \tag{4.4}$$

Note that as an immediate consequence of (1.1) and (4.1) we have

$$f^{\#}(x) \le 2Mf(x) \le 2\|f\|_{\infty}. \tag{4.5}$$

Together with (4.2), this shows that L^{∞} is continuously embedded in BMO.

One reason for the importance of BMO is that it arises as (essentially) the range of certain singular integral operators, such as the Hilbert or Riesz transforms, acting on L^{∞}. Consequently, the interpolation properties of BMO are also of much interest. Now while BMO contains L^{∞}, the fundamental John-Nirenberg lemma (of which we give an alternate proof in Corollary 4.6) shows that it is only "slightly" larger than L^{∞}. Hence, we might expect the (θ,q)-interpolation spaces for the pairs (L^1,L^{∞}) and (L^1,BMO) to be the same. This result, namely

$$(L^1,L^{\infty})_{\theta,q} = L^{pq} = (L^1,BMO)_{\theta,q}, \quad \theta = 1 - 1/p, \tag{4.6}$$

valid for $0 < \theta < 1$ and $0 < q \le \infty$, is due to Hanks [12].

While the first equivalence in (4.6) is, of course, well-known [4, p.186], it will nevertheless be instructive to examine its proof. Recall [4, p.167] that f belongs to the interpolation space $(X_0,X_1)_{\theta,q}$ if $(\int_0^{\infty}[t^{-\theta}K(f;t)]^q dt/t)^{1/q}$

is finite. Since $K(f;t;L^1,L^\infty) = tf^{**}(t)$ [4, p.184], we see that the norms in $(L^1,L^\infty)_{\theta,q}$ and L^{pq} are

$$\left(\int_0^\infty [t^{1/p}f^{**}(t)]^q dt/t\right)^{1/q}, \quad \left(\int_0^\infty [t^{1/p}f^{*}(t)]^q dt/t\right)^{1/q}, \quad \theta = 1-1/p,$$

respectively. That the first dominates the second is clear from the "direct" inequality

$$f^* \le f^{**}.$$

But in the other direction these quantities are related by the "inverse" weak-type inequality

$$f^{**}(t) \le \frac{1}{t}\int_0^t f^*(s)ds.$$

Hence, by applying L^{pq}-norms, we see that the two spaces coincide whenever the averaging operator is bounded on L^{pq}, that is, whenever $p > 1$.

While this is an admittedly trivial example of a weak-type inequality, it does point out a direction to be followed in establishing the more complex second equivalence in (4.6). Indeed, the inclusion $(L^1,L^\infty)_{\theta,q} \subseteq (L^1,BMO)_{\theta,q}$ follows immediately from the direct inequality

$$K(f;t;L^1,BMO) \le 2K(f;t;L^1,L^\infty), \tag{4.7}$$

which itself follows from the definition of the K-functional and the obvious estimate $\|f\|_{BMO} \le 2\|f\|_\infty$. By analogy with the previous situation, what is needed to establish the reverse inclusion is a weak-type inequality, inverse to (4.7). This we shall do in Corollary 4.4. In the process, we shall establish a weak-type inequality which is inverse to the direct inequality $(f^\#)^* \le 2(Mf)^*$, embodied in (4.5). This is the content of the next theorem.

THEOREM 4.1. If $f \in L^1 + L^\infty$, then

$$(Mf)^*(t) \le c\int_t^\infty (f^\#)^*(s)\,\frac{ds}{s} + (Mf)^*(+\infty), \qquad 0 < t < \infty. \tag{4.8}$$

Proof. The open set $\Omega = \{f^\# > (f^\#)^*(2t)\} \cup \{Mf > (Mf)^*(2t)\}$ has measure

$$|\Omega| \le 4t. \tag{4.9}$$

Applying Lemma 1.1, we obtain a covering of Ω by cubes $Q_j, j = 1,2,\ldots$, with pairwise disjoint interiors, such that each Q_j has nonempty intersection with Ω^c, and $\sum_j |Q_j| \le 2^n|\Omega|$. If $F = (\bigcup_j Q_j)^c$, define

$$g = \sum_j [f - Q_j(f)]\chi_{Q_j}, \quad h = \sum_j Q_j(f)\chi_{Q_j} + f\chi_F.$$

Then $f = g + h$, and so

$$(Mf)^*(t) \le (Mg)^*(t) + \|h\|_\infty \le 3^n t^{-1}\|g\|_1 + \|h\|_\infty, \tag{4.10}$$

since M is of weak-type (1,1). For any point $x_j \in Q_j \cap \Omega^c$, we have $Q_j(|f-Q_j(f)|) \le f^{\#}(x_j) \le (f^{\#})^*(2t)$. Hence, by (4.9) and Lemma 1.1(c),

$$\|g\|_1 = \sum_j |Q_j| Q_j(|f-Q_j(f)|) \le \sum_j |Q_j| (f^{\#})^*(2t) \le 4\cdot 2^n t (f^{\#})^*(2t). \tag{4.11}$$

On the other hand, $Q_j(|f|) \le Mf(x_j) \le (Mf)^*(2t)$, and so

$$\|h\|_\infty = \max\{\|f\chi_F\|_\infty, \sup_j Q_j(|f|)\} \le \max\{\|(Mf)\chi_F\|_\infty, (Mf)^*(2t)\} \le (Mf)^*(2t). \tag{4.12}$$

Combining inequalities (4.10), (4.11), and (4.12), we obtain

$$(Mf)^*(t) \le 4\cdot 6^n (f^{\#})^*(2t) + (Mf)^*(2t). \tag{4.13}$$

Iteration of (4.13) N times, with $2^k t$ replacing $2^{k-1}t$, $k = 1,2,\dots,N$, gives

$$(Mf)^*(t) \le 4\cdot 6^n \sum_{k=1}^{N} (f^{\#})^*(2^k t) + (Mf)^*(2^N t).$$

But since $(f^{\#})^*$ is nonincreasing, the sum can be estimated in terms of an integral:

$$(Mf)^*(t) \le 8.6^n \int_t^{2^N t} (f^{\#})^*(s)\frac{ds}{s} + (Mf)^*(2^N t). \tag{4.14}$$

Inequality (4.8) now follows by letting N tend to infinity.

COROLLARY 4.2. If $f \in L^1 + L^\infty$, then

$$f^{**}(t) \le c\{\int_t^\infty (f^{\#})^*(s)\frac{ds}{s} + f^{**}(+\infty)\}, \qquad 0 < t < \infty. \tag{4.15}$$

Moreover, if $f^{**}(+\infty) = 0$ (in particular, if $f \in L^r$ for any $r < \infty$), then there are constants c_1 and c_2 independent of f such that

$$c_1\|f\|_{pq} \le \|f^{\#}\|_{pq} \le c_2\|f\|_{pq}, \qquad 1 < p < \infty,\ 0 < q < \infty. \tag{4.16}$$

Proof. The inequality (4.15) follows directly from (4.8) and Theorem 1.3. We introduce the notation P and P' for the averaging operators

$$P(f^*)(t) = \frac{1}{t}\int_0^t f^*(s)ds \tag{4.17}$$

and

$$P'(f^*)(t) = \int_t^\infty f^*(s)\frac{ds}{s}. \tag{4.18}$$

From (4.5) and (1.4) we have $(f^{\#})^*(t) \le cf^{**}(t)$. Hence, applying L^{pq}-norms and using Hardy's inequality for P, we obtain $\|f^{\#}\|_{pq} \le c\|f\|_{pq}$, provided $1 < p \le \infty$. In the other direction, we note that $f^{**}(+\infty) = 0$, by hypothesis. Hence, by (4.15), $f^*(t) \le f^{**}(t) \le cP'(f^{\#})^*(t)$. Applying L^{pq}-norms and using

Hardy's inequality for P', we now obtain $\|f\|_{pq} \le c\|f^{\#}\|_{pq}$, provided $1 \le p < \infty$. This completes the proof.

It will be shown in §6 that $K(f;t;L^1,BMO) \sim t(f^{\#})^*(t)$. All that is needed at the present time is the "easy" half of this estimate:

LEMMA 4.3. If $f \in L^1 + BMO$, then

$$t(f^{\#})^*(t) \le cK(f;t;L^1,BMO), \qquad 0 < t < \infty. \tag{4.19}$$

Proof. If $f = g + h$, with $g \in L^1$ and $h \in BMO$, then $f^{\#} \le g^{\#} + h^{\#} \le g^{\#} + \|h\|_{BMO}$, so $t(f^{\#})^*(t) \le t(g^{\#})^*(t) + t\|h\|_{BMO}$. Using (4.5) and the weak (1,1) estimate for M, we obtain $t(f^{\#})^*(t) \le 2\cdot 3^n\|g\|_1 + t\|h\|_{BMO}$. Taking the infimum over all decompositions $f = g + h$, we get the desired inequality (4.19).

Now we are in a position to establish the weak-type inequality which is inverse to (4.7), and hence to identify the interpolation spaces $(L^1,BMO)_{\theta,q}$.

COROLLARY 4.4. The identity operator is of generalized weak-type $[1,1;\infty,\infty)$ with respect to the pairs (L^1,BMO) and (L^1,L^∞), that is,

$$\frac{K(f;t;L^1,L^\infty)}{t} \le c\left(\int_t^\infty \frac{K(f,s;L^1,BMO)}{s}\frac{ds}{s} + \lim_{N\to\infty}\frac{K(f;N;L^1,L^\infty)}{N}\right). \tag{4.20}$$

Consequently,

$$(L^1,L^\infty)_{\theta,q} = (L^1,BMO)_{\theta,q}, \qquad 0 < \theta < 1, \quad 0 < q \le \infty. \tag{4.21}$$

Proof. Inequality (4.20) follows from (4.15), (4.19) and the fact that $K(f;t;L^1,L^\infty) = tf^{**}(t)$. One of the inclusions in (4.21) is given by (4.7) so we need only show that $(L^1,BMO)_{\theta,q} \subseteq (L^1,L^\infty)_{\theta,q}$. First suppose $f \in L^1 \cap BMO$. Then $f^{**}(+\infty) = 0$ and so the last term in (4.20) vanishes. Applying the L^{pq}-norm (with $\theta = 1-1/p$) to (4.20), and using Hardy's inequality for the operator P', we obtain

$$\|f\|_{(L^1,L^\infty)_{\theta,q}} \le c\|f\|_{(L^1,BMO)_{\theta,q}}, \qquad f \in L^1 \cap BMO.$$

But $L^1 \cap BMO$ is dense in the interpolation space $(L^1,BMO)_{\theta,q}$ [4, Chapter 3] for $0 < q < \infty$, so the inequality persists for all $f \in (L^1,BMO)_{\theta,q}$. This establishes (4.21) for $q < \infty$. The remaining case $q = \infty$ is then settled by using the reiteration theorem [4, p.177].

Now let us show how the John-Nirenberg lemma can be derived from the weak-type inequality (4.15). Actually, what we need is a "local" version of (4.15) relating to a fixed cube Q. At the same time, however, we may as well establish a more general inequality which will also give us basic information concerning the $L^{p,\lambda}$ spaces [22]. For a fixed cube Q, and $0 \le \alpha \le 1$, let

$$f_Q^{\#,\alpha}(x) = \sup_{\substack{Q' \subseteq Q \\ x \in Q'}} \{|Q'|^{-\alpha/n} Q'(|f-Q'(f)|)\}\chi_Q(x),$$

and

$$(M_Q f)(x) = \sup_{\substack{Q' \subseteq Q \\ x \in Q'}} \{Q'(|f|)\}\chi_Q(x).$$

The local version of (4.15) is as follows.

LEMMA 4.5. Suppose Q is a fixed cube in R^n, possibly R^n itself, and let $0 \le \alpha \le 1$. Then there is a constant c, depending only on the dimension n, such that for each locally integrable function f on R^n,

$$(4.22) \qquad (f\chi_Q)^{**}(t) \le c\left\{\int_t^{|Q|} (f_Q^{\#,\alpha})^*(s)s^{\alpha/n}\,\frac{ds}{s} + Q(|f|)\right\}, \qquad 0 < t \le \frac{|Q|}{2}.$$

Before proceeding with the proof of Lemma 4.5, let us examine some of its consequences. Note that (4.15) is in fact the special case of (4.22) corresponding to $Q = R^n$ and $\alpha = 0$. In the resulting inequalities (4.16), which are due originally to Fefferman-Stein [10], the non-trivial assertion is that $f^{\#} \in L^p$ implies $f \in L^p$, for $1 < p < \infty$. The John-Nirenberg lemma can be regarded as the limiting case $p = \infty$ of this result: it asserts that if $f^{\#} \in L^{\infty}$ (i.e., $f \in$ BMO), then f is "locally" exponentially integrable.

COROLLARY 4.6 (John-Nirenberg [15]). Suppose f is locally integrable on R^n and let Q be a fixed cube in R^n. Then

$$(4.23) \qquad [(f-Q(f))\chi_Q]^{**}(t) \le c\int_t^{|Q|} (f_Q^{\#,0})^*(s)\,\frac{ds}{s}, \qquad 0 < t < \frac{|Q|}{2}.$$

If $f \in$ BMO, then

$$K(Q) = \|f_Q^{\#,0}\|_{\infty} = \sup_{Q' \subseteq Q} Q'(|f-Q'(f)|) < \infty,$$

and

$$(4.24) \qquad [(f-Q(f))\chi_Q]^*(t) \le cK(Q)\log^+\left(\frac{2|Q|}{t}\right), \qquad 0 < t < \infty.$$

In terms of the distribution function, (4.24) asserts

$$(4.25) \qquad |\{x \in Q : |f(x) - Q(f)| > \lambda\}| \le 2|Q|\exp\left(\frac{-\lambda}{cK(Q)}\right), \qquad 0 < \lambda < \infty.$$

Proof. The inequality (4.23) follows by applying (4.22) to the function $f - Q(f)$. The resulting constant term $Q(|f - Q(f)|)$ does not exceed $(f_Q^{\#,0})^*(|Q|-)$. Indeed, $Q(|f - Q(f)|) \le f_Q^{\#,0}(x)$, for every $x \in Q$, that is, on a set of measure $|Q|$. Since a function and its decreasing rearrangement are equi-distributed, the result follows. But now $(f_Q^{\#,0})^*(|Q|-)$ is dominated by a constant multiple of the integral in (4.22), since $t \le |Q|/2$. This establishes (4.23). The inequality (4.24) is an easy consequence since $g^* \le g^{**}$ and $[(f - Q(f))\chi_Q]^*(t) = 0$ for $t > |Q|$. Finally, (4.25) is an equivalent re-statement of (4.24) since the decreasing rearrangement and the distribution function are mutually inverse.

The space $\mathcal{L}^{p,\lambda}$ (cf. [22]) consists of those locally integrable functions f on R^n for which the norm (modulo constants)

$$\|f\|_{\mathcal{L}^{p,\lambda}} = \sup_Q \{|Q|^{1-\lambda/n} Q(|f - Q(f)|^p)\}^{1/p}, \qquad 1 \le p < \infty,$$

is finite. Clearly, $\mathcal{L}^{1,n} = \mathrm{BMO}$ and so, by analogy, the space $\mathcal{L}^{p,n}$, $1 \le p \le \infty$, is often denoted by BMO(p).

COROLLARY 4.7 [15]. For $1 \le p < \infty$, $\mathrm{BMO}(p) = \mathrm{BMO}$, with equivalent norms.

Proof. The inclusion $\mathrm{BMO}(p) \subseteq \mathrm{BMO}$ results directly from (4.24) and the fact that $\int_2^\infty (\log u)^p du/u^2 < \infty$. The opposite containment is an immediate consequence of Hölder's inequality.

COROLLARY 4.8 (Campanato [6], Meyers [17]). Suppose $1 \le p < \infty$ and $n < \lambda \le n + p$. Then

$$\mathcal{L}^{p,\lambda} = \mathrm{Lip}(\alpha), \qquad \alpha = (\lambda - n)/p,$$

with equivalent norms.

Proof. Note that $0 < \alpha \le 1$. It is easy to see that $\mathrm{Lip}(\alpha) \subseteq \mathcal{L}^{p,\lambda}$: if $f \in \mathrm{Lip}(\alpha)$, then for any $x \in Q$,

$$|f(x) - Q(f)| \le \frac{1}{|Q|}\left|\int_Q [f(x) - f(y)]dy\right| \le c\|f\|_{\mathrm{Lip}(\alpha)}|Q|^{\alpha/n},$$

from which the finiteness of the $\mathcal{L}^{p,\lambda}$-norm follows. In the other direction, Holder's inequality gives

$$(4.26) \qquad f_Q^{\#,\alpha}(x) \le \sup_{\substack{Q' \subseteq Q \\ x \in \overline{Q'}}} \{|Q'|^{-\alpha/n} Q'(|f - Q'(f)|^p)^{1/p}\} \le \|f\|_{\mathcal{L}^{p,\lambda}}, \quad x \in Q,$$

that is, $f_Q^{\#,\alpha}$ is bounded on Q by $\|f\|_{\mathcal{L}^{p,\lambda}}$. Using this estimate in (4.22) (applied to $f - f_Q$ rather than f, and with $t \to 0$), we find that

$$\|(f - f_Q)\chi_Q\|_\infty \le c\|f\|_{\mathcal{L}^{p,\lambda}}|Q|^{\alpha/n}$$

Hence

$$|f(x) - f(y)| \le 2c\|f\|_{L^{p,\lambda}} |Q|^{\alpha/n}, \qquad x,y \in Q,$$

and so $f \in \mathrm{Lip}(\alpha)$. Note that if $K_\alpha(Q)$ denotes the supremum of the left-hand side of (4.26), then the same argument produces the last estimate with $\|f\|_{L^{p,\lambda}}$ replaced by $K_\alpha(Q)$. Hence, if $K_\alpha(Q) \to 0$ as $|Q| \to 0$, we see that $f \in \mathrm{lip}\,(\alpha)$.

As a final application of (4.22), we complement Corollary 4.7 by showing that $BMO(p) = BMO$, for $0 < p < 1$ (cf. [12]). Let

$$f_Q^p(x) = \sup_{\substack{Q' \subseteq Q \\ x \in \overline{Q}'}} \{\inf_{c_{Q'}} Q'(|f - c_{Q'}|^p)^{1/p}\} \chi_Q(x). \tag{4.27}$$

The space $BMO(p)$ consists of those f for which $\|f^p_{R^n}\|_\infty < \infty$.

COROLLARY 4.9. If $0 < p < 1$, then

$$BMO(p) = BMO, \tag{4.28}$$

and

$$\frac{K(f;t;L^p,L^\infty)}{t} \le c\{\int_t^\infty [\frac{K(f;s;L^p,BMO)}{s}]^p \frac{ds}{s} + (|f|^p)^{**}(+\infty)\}^{1/p}. \tag{4.29}$$

Proof. Hölder's inequality gives

$$(|f|^p)_Q^{\#,0}(x) \le 2[f_Q^p(x)]^p, \tag{4.30}$$

since $||f(x)|^p - |c|^p| \le |f(x) - c|^p$ for $0 < p < 1$. Using (4.30) in the inequality (4.23) (applied to $|f|^p$ instead of f) we obtain

$$(|f|^p\chi_Q)^{**}(t) \le c\{\int_t^{|Q|} (f_Q^p)^*(s)^p \frac{ds}{s} + Q(|f|^p)\}, \qquad 0 < t < \frac{|Q|}{2}. \tag{4.31}$$

Now replace f by $f-c_Q$ and estimate f_Q^p from above by $f^p_{R^n}$ to get

$$[(f-c_Q)\chi_Q]^*(t)^p \le c\,\|f\|^p_{BMO(p)} \log^+(\frac{2|Q|}{t}).$$

Taking p-th roots and integrating from 0 to $|Q|$ we find that $\|f\|_{BMO} \le c\|f\|_{BMO(p)}$. The reverse inequality follows immediately from Hölder's inequality. This establishes (4.28).

The inequality (4.31) with $Q = R^n$ gives

$$\{\frac{1}{t}\int_0^t f^*(s)^p ds\} \le c\{\int_t^\infty (f^p_{R^n})^*(s)^p \frac{ds}{s} + (|f|^{p})^{**}(+\infty)\}, \qquad 0 < t < \infty. \tag{4.32}$$

But $K(f;t;L^p,L^\infty) \sim \{\int_0^{t^p} f^*(s)^p ds\}^{1/p}$, so taking p-th roots in (4.32) and changing variables $t \to t^p$ we get

$$\frac{K(f;t;L^p,L^\infty)}{t} \le c\{\int_{t^p}^\infty (f^p_{R^n})^*(s)^p \frac{ds}{s} + (|f|^p)^{**}(+\infty)\}^{1/p}.$$

This, together with the estimate

$$(4.33)\qquad (f^p_{R^n})^*(t^p) \le c\,\frac{K(f;t;L^p,\mathrm{BMO})}{t},$$

establishes (4.29). The proof of (4.33) is exactly the same as that of (4.19), but uses the equivalence of BMO and BMO(p), $0 < p < 1$, as well as the simple fact that $g^p_{R^n}(x) \le M(|g|^p)^{1/p}(x)$.

We close this section with a proof of Lemma 4.5. Since it closely follows the proof of Theorem 4.1, we shall be brief.

Proof of Lemma 4.5. If $0 < t \le |Q|/2$, then the set

$$\Omega_Q = \{x : f^{\#,\alpha}_Q(x) > (f^{\#,\alpha}_Q)^*(2t) \text{ or } (M_Q f)(x) > (M_Q f)^*(2t)\}$$

has measure not exceeding 4t. Since $\Omega_Q \subseteq Q$, the remark following Lemma 1.1 shows that there is a covering $\{Q_j\}$ of Ω_Q by cubes Q_j, with pairwise disjoint interiors, such that each Q_j is contained in Q and has nonempty intersection with $Q-\Omega_Q$, and $\Sigma|Q_j| \le 2^n\,|\Omega_Q|$. If $f = g + h$, then

$$(4.34)\qquad (f\chi_Q)^{**}(t) \le t^{-1}\|g\chi_Q\|_1 + \|h\chi_Q\|_\infty$$

and

$$(4.35)\qquad (M_Q f)^*(t) \le 3^n t^{-1}\|g\chi_Q\|_1 + \|h\chi_Q\|_\infty ,$$

since M is of weak type (1,1). With $F = Q-(\cup Q_j)$ and

$$g = \sum_j [f-Q_j(f)]\chi_{Q_j}, \qquad h = \sum_j Q_j(f)\chi_{Q_j} + f\chi_F ,$$

we have

$$(4.36)\qquad \|g\|_1 = \sum_j |Q_j|^{1+\alpha/n}\{|Q_j|^{-\alpha/n}Q_j(|f-Q_j(f)|)\}$$

$$\le (\sum_j |Q_j|)^{1+\alpha/n}(f^{\#,\alpha}_Q)^*(t) \le ct^{1+\alpha/n}(f^{\#,\alpha}_Q)^*(2t),$$

and

$$(4.37)\qquad \|h\|_\infty \le \max\{\sup_j Q_j(|f|),\ \|f\chi_F\|_\infty\} \le (M_Q f)^*(2t).$$

Using (4.36) and (4.37) in the inequalities (4.34) and (4.35), we obtain

$$(4.38)\qquad (f\chi_Q)^{**}(t) \le ct^{\alpha/n}(f^{\#,\alpha}_Q)^*(2t) + (M_Q f)^*(2t)$$

and

$$(4.39)\qquad (M_Q f)^*(t) \le ct^{\alpha/n}(f^{\#,\alpha}_Q)^*(2t) + (M_Q f)^*(2t).$$

Let $N \ge 1$ be the unique integer satisfying $2^N t \le |Q| < 2^{N+1}t$. Using (4.39) to estimate the right-hand side of (4.38), and iterating this process (N-1) times, we find

$$(f\chi_Q)^{**}(t) \le \sum_{k=1}^{N}(2^k t)^{\alpha/n}(f_Q^{\#,\alpha})^*(2^k t) + (M_Q f)^*(2^N t). \tag{4.40}$$

The sum can be estimated by the integral $\int_t^{2^N t} s^{\alpha/n}(f_Q^{\#,\alpha})^*(s)\,\frac{ds}{s}$, and the constant term by

$$(M_Q f)^*(2^N t) \le [M(f\chi_Q)]^*(2^N t) \le 3^n(2^N t)^{-1}\|f\chi_Q\|_1 \le 2.3^n Q(|f|).$$

This establishes (4.22) and hence completes the proof.

§5. INTERPOLATION FOR (H^1,L^∞) AND (H^1,BMO). Rivière-Sagher [24] showed that the (θ,q)-interpolation spaces for the pair (H^1,L^∞) are the same as those for (L^1,L^∞), namely

$$(H^1,L^\infty)_{\theta,q} = L^{pq}, \qquad 0 < \theta < 1, \quad 0 < q \le \infty, \tag{5.1}$$

where $\theta = 1-1/p$. Subsequently, Fefferman-Rivière-Sagher [9] used the newly-developed methods of the Fefferman-Stein real-variable H^p-theory [10] to determine the K-functional for H^p and L^∞. Thus

$$K(f;t;H^p,L^\infty) \sim \{\int_0^{t^p}(\mathcal{M}f)^*(s)^p ds\}^{1/p},$$

where $\mathcal{M}f$ is the grand maximal function of f. It follows directly from the weak-type (1,1) and strong-type (∞,∞) properties of $\mathcal{M}$ that

$$\frac{K(f;t;H^1,L^\infty)}{t} \le c\,\frac{1}{t}\int_0^t f^{**}(s)ds \le c\,\frac{1}{t}\int_0^t \frac{K(f;s;L^1,L^\infty)}{s}\frac{ds}{s}. \tag{5.2}$$

This inequality asserts that the identity operator is of generalized weak type $(1,1;\infty,\infty)$ for the pairs (L^1,L^∞) and (H^1,L^∞). This weak-type inequality is the inverse of the direct estimate

$$\frac{K(f;t;L^1,L^\infty)}{t} \le c\,\frac{K(f;t;H^1,L^\infty)}{t}, \tag{5.3}$$

which holds because $\|f\|_{L^1} \le \|f\|_{H^1}$. Note that (5.3) and (5.2) (via Hardy's inequality) imply (5.1).

In this section, we present a proof of (5.2) using only the weak-type (1,1) and strong-type (2,2) properties of the Riesz transforms. This gives a good illustration, as does the Rivière-Sagher result, of the fact that the (θ,q) interpolation spaces, for $0 < \theta < 1$, can often be determined without specific knowledge of the K-functional. Such knowledge is usually required, however, in dealing with the situation at the endpoints $\theta = 0$ or $\theta = 1$. Furthermore, once (5.2) has been established, we can combine it with the weak-type inequalities from the previous section and thereby determine the (θ,q) interpolation spaces $(0 < \theta < 1)$ for H^1 and BMO (which, once again, are the Lorentz spaces L^{pq} [12]). Finally, we present a basic inequality for the Hilbert transform (Corollary 5.5) which, besides storing information relative to H^1 and BMO, implies the rearrangement-invariant inequality (2.1) of O'Neil-Weiss.

DEFINITION 5.1 [10]. The Hardy space $H^1(R^n)$ consists of those L^1-functions f whose Riesz transforms

$$R_j f(x) = c_n \int_{R^n} f(x-y) \frac{y_j}{|y|^{n+1}}\, dy, \qquad j = 1,2,\dots,n,$$

also belong to $L^1(R^n)$. H^1 is a Banach space when given the norm

$$\|f\|_{H^1} = \|f\|_{L^1} + \sum_{j=1}^{n} \|R_j f\|_{L^1} .$$

The right-hand side of (5.2) is finite for all t if and only if $\int_0^1 f^{**}(t)\, dt < \infty$. Now it is not hard to show (cf. [2, Corollary 10.2], for example) that this condition is equivalent to membership of f in the space $L \log L + L^\infty$, that is, f is representable as a sum $f = g + h$, with $h \in L^\infty$ and

$$\int_{R^n} |g(x)| \log\,(2 + |g(x)|)dx < \infty .$$

Indeed, the functional $\int_0^1 f^{**}(t)dt$ defines an equivalent norm on $L \log L + L^\infty$.

The elementary argument used in the proof of this equivalence shows, in particular, that

$$\int_\Omega |g(x)|\log(2 + |g(x)|)dx \le \|g\|_{L^1}\log(2 + \|g\|_{L^1}) + \int_0^1 g^{**}(t)dt, \tag{5.4}$$

for any locally integrable function g whose support Ω has measure at most one. We shall need to use this inequality in the proof of the next theorem.

THEOREM 5.2. If $f \in L \log L + L^\infty$, then

$$t^{-1}K(f;t;H^1,L^\infty) \le ct^{-1}\int_0^t f^{**}(s)ds, \qquad 0 < t < \infty. \tag{5.5}$$

Proof. Since the Riesz transforms commute with dilations, it is enough to establish (5.5) for $t = 1$. Also, by homogeneity, we can assume

$$\int_0^1 f^{**}(s)ds \le 1. \tag{5.6}$$

Hence, it will suffice to find a constant c, depending only on the dimension n, such that

$$K(f;1;H^1;L^\infty) \le c, \tag{5.7}$$

for all f satisfying (5.6). This will be done by producing $g \in H^1$ and $h \in L^\infty$ with $f = g + h$ and

$$\|g\|_{H^1} + \|h\|_{L^\infty} \le c. \tag{5.8}$$

Now the open set $\Omega = \{(Mf)(x) > (Mf)^*(1)\}$ has measure at most one. Let $\{Q_k\}$ be a Whitney decomposition [27, Chapter 1] of Ω : the cubes Q_k therefore have union Ω, have pairwise disjoint interiors, and satisfy $(\beta_n \cdot Q_k) \cap \Omega^c \neq \phi$, where $\beta_n = 10n^{1/2}$ (here $\beta_n \cdot Q$ denotes the cube concentric with Q, having β_n-times the diameter of Q). If $F = \Omega^c$, define

$$g = \sum_k [f - Q_k(f)]\chi_{Q_k}, \qquad h = \sum Q_k(f)\chi_{Q_k} + f\chi_F.$$

Since $\beta_n \cdot Q_k$ meets F, we have

$$|Q_k(f)| \le Q_k(|f|) \le \beta_n^n(\beta_n \cdot Q_k)(|f|) \le \beta_n^n\,(Mf)^*(1).$$

Hence, by (1.4) and (5.6),

$$|Q_k(f)| \le (3\beta_n)^n \int_0^1 f^*(s)ds \le (3\beta_n)^n \int_0^1 f^{**}(s)ds \le (3\beta_n)^n . \tag{5.9}$$

Similarly,

$$\|f\chi_F\|_\infty \le \|(Mf)\chi_F\|_\infty \le (Mf)^*(1) \le 3^n, \tag{5.10}$$

and so (5.9) and (5.10) combine to give

$$\|h\|_\infty \le (3\beta_n)^n. \tag{5.11}$$

To estimate the H^1-norm of g, we first note that

$$g^{**}(s) \le cf^{**}(s), \qquad 0 < s < 1. \tag{5.12}$$

Indeed, if E is any subset of Ω with $|E| = t$, then

$$\int_E |g| \le \sum_k \int_{E\cap Q_k} |f - Q_k(f)| \le \sum_k \int_{E\cap Q_k} |f| + \sum_k |E\cap Q_k|\beta_n^n(\beta_n \cdot Q_k)(|f|).$$

Since each cube $(\beta_n \cdot Q_k)$ meets Ω^c, we have, using (1.4), $(\beta_n \cdot Q_k)(|f|) \le (Mf)^*(1) \le 3^n f^{**}(1)$. Hence

$$\int_E |g| \le \int_E |f| + |E|(3\beta_n)^n f^{**}(1) \le \int_0^t f^*(s)ds + t(3\beta_n)^n f^{**}(t).$$

Taking the supremum over all sets E of measure t, we obtain (5.12) with $c = (3\beta_n)^n + 1$.

Next we use the following classical estimate for the Riesz transforms of a function b with mean value zero and support in a cube Q:

$$\|R_j b\|_{L^1} \le c\left(\int_Q |b(x)|\log(2 + |b(x)|)dx + |Q|\right), \qquad j = 1,2,\dots,n. \tag{5.13}$$

The proof uses a standard Marcinkiewicz-type decomposition argument (cf. [8, vol. II, p.166]) involving the weak-type (1,1) and strong-type (2,2) properties of the Riesz transforms. We omit the details.

Applying (5.13) to the functions $g\chi_{Q_k}$, we find

$$\|g\|_{H^1} \le \|g\|_{L^1} + \sum_{j=1}^{n} \sum_k \|R_j(g\chi_{Q_k})\|_{L^1}$$

$$\le \|g\|_{L^1} + c \sum_k \{\int_{Q_k} |g(x)| \log(2 + |g(x)|)dx + |Q_k|\}$$

$$= \|g\|_{L^1} + c\{\int_\Omega |g(x)| \log(2 + |g(x)|)\, dx + |\Omega|\}.$$

Since $|\Omega| \le 1$, we can apply (5.4) to obtain

$$\|g\|_{H_1} \le c\{\|g\|_{L^1} \log(2 + \|g\|_{L^1}) + \int_0^1 g^{**}(t)dt\ .$$

Hence, using the estimate $\|g\|_{L^1} = \int_0^1 g^*(t)dt \le \int_0^1 g^{**}(t)dt$, we see from (5.12) and (5.6) that $\|g\|_{H^1}$ is bounded from above by a constant (depending only on n). This completes the proof.

For the corresponding function spaces on the unit circle, Zygmund's theorem (Corollary 2.1(b)) asserts that $L \log L \subseteq H^1$. This result is decidedly false in R^n because, for example, $H^1(R^n)$ contains no positive functions. There is however the following natural analogue in R^n of Zygmund's theorem, which follows directly from (5.5) with $t = 1$.

<u>COROLLARY 5.3.</u> $\qquad L \log L + L^\infty \subseteq H^1 + L^\infty$.

As we remarked above, an immediate consequence of the previous theorem and Corollary 4.4 is that

$$(5.14) \qquad (H^1, L^\infty)_{\theta,q} = L^{pq} = (L^1, BMO)_{\theta,q}, \qquad 0 < \theta < 1, \quad 0 < q \le \infty,$$

where $\theta = 1-1/p$. But the embeddings $L^\infty \subseteq BMO$ and $H^1 \subseteq L^1$ imply

$$(H^1, L^\infty)_{\theta,q} \subseteq (H^1, BMO)_{\theta,q} \subseteq (L^1, BMO)_{\theta,q}.$$

Together with (5.14), this produces the following result, due to Hanks [12].

<u>COROLLARY 5.4.</u> If $\theta = 1-1/p$, then

$$(H^1, BMO)_{\theta,q} = L^{pq}, \qquad 0 < \theta < 1, \quad 0 < q \le \infty .$$

For any operator T carrying H^1 into L^1 and L^∞ into BMO, it follows easily (cf. [4, p.180]) from the definition of the K-functional that

$$K(Tf;t;L^1,BMO) \le cK(f;t;H^1,L^\infty).$$

Hence, using the Fefferman-Rivière-Sagher result and Lemma 4.3, we obtain, for the Riesz transforms in particular, the following interesting inequality.

COROLLARY 5.5. If $f \in H^1 + L^\infty$, then the Riesz transforms satisfy

$$[(R_j f)^{\#}]^*(t) \le c(Mf)^{**}(t), \qquad 0 < t < \infty, \quad j = 1,2,\dots,n, \tag{5.15}$$

where M is the grand maximal function.

Let f be a simple function so, in particular, $(R_j f)^{**}(+\infty) = 0$. Applying (4.15) and (5.15) to $R_j f$ we obtain

$$\int_0^1 (R_j f)^*(s)ds \le c\int_1^\infty [(R_j f)^{\#}]^*(s)\,\frac{ds}{s} \le c\int_1^\infty (Mf)^{**}(s)\,\frac{ds}{s}.$$

Hence (5.5) and a change in the order of integration gives

$$\int_0^1 (R_j f)^*(s)ds \le c\{\int_0^1 f^{**}(u)du + \int_1^\infty f^{**}(u)\,\frac{du}{u}\}. \tag{5.16}$$

This, in the notation of [2], simply asserts the boundedness (on the simple functions) of R_j from the space $L \log L + L^{\infty 1}$ into $L^1 + L^\infty$. Since the simple functions are dense in $L \log L + L^{\infty 1}$, the inequality (5.16) persists for all f. A simple dilation argument and the fact that the Riesz transforms commute with dilations now gives the basic rearrangement-invariant inequality for the Riesz transforms, due to O'Neil-Weiss [21] (which we originally discussed in (2.1) in the context of the unit circle).

COROLLARY 5.6. If $f \in L \log L + L^{\infty 1}$, then the Riesz transforms satisfy

$$(R_j f)^{**}(t) \le c\{\frac{1}{t}\int_0^t f^{**}(s)ds + \int_t^\infty f^{**}(s)\,\frac{ds}{s}\}, \quad 0 < t < \infty,\ j=1,2,\dots,n. \tag{5.17}$$

The fundamental inequality (5.15) thus stores not only the information $R_j : H^1 \to L^1$ and $R_j : L^\infty \to BMO$, but also, via (5.17), the rearrangement-invariant behavior of the Riesz transforms. Since a potentially "deeper" inequality could result from encoding the information $R_j : H^1 \to H^1$ and $R_j : BMO \to BMO$, it would seem to be a problem of some interest to describe in concrete terms the K-functional for the pair (H^1, BMO).

§6. THE K-FUNCTIONAL FOR L^1 AND BMO. In this final section, we show that the K-functional for L^1 and BMO can be identified with $t(f^{\#})^*(t)$.

THEOREM 6.1. There are constants c_1 and c_2, depending only on the dimension n, such that for any $f \in (L^1 + BMO)(R^n)$,

$$c_1 t(f^{\#})^*(t) \le K(f;t;L^1,BMO) \le c_2 t(f^{\#})^*(t), \qquad 0 < t < \infty . \tag{6.1}$$

Proof. The first inequality in (6.1) was established in Lemma 4.3. In order to prove the second inequality, we must exhibit $g \in L^1$ and $h \in BMO$ such that $f = g + h$ and

$$\|g\|_{L^1} + t\|h\|_{BMO} \le c_2 t(f^{\#})^*(t). \tag{6.2}$$

With t fixed, the open set $\Omega = \{x : f^{\#}(x) > (f^{\#})^*(t)\}$ has measure not exceeding t. Let $F = \Omega^c$ and let $\{Q_j\}$ be a Whitney covering of Ω (cf. [27]). Thus, the cubes Q_j are dyadic and satisfy

$$\Omega = \cup_j Q_j ; \tag{6.3}$$

$$|Q_j \cap Q_k| = 0, \qquad \text{unless } j = k; \tag{6.4}$$

$$\text{diam } (Q_j) \le \text{dist } (Q_j,F) \le 4 \text{ diam } (Q_j), \qquad j = 1,2,\ldots \tag{6.5}$$

If $\alpha\cdot Q$ $(\alpha > 0)$ denotes the cube concentric with Q but having α-times the diameter, then (6.5) shows that with $\beta = \beta(n) = 10n^{1/2}$,

$$(\beta\cdot Q_j) \cap F \ne \phi, \qquad j = 1,2,\ldots \tag{6.6}$$

We shall denote by $\overline{Q}_j$ the cube $\beta\cdot Q_j$.

The decomposition to be used in (6.2) is given by

$$g(x) = \sum_j (f(x) - Q_j(f))\chi_{Q_j}(x) \tag{6.7}$$

and

$$h(x) = \sum_j Q_j(f)\chi_{Q_j}(x) + f(x)\chi_F(x). \tag{6.8}$$

For any cube Q (so $\overline{Q} = \beta\cdot Q$), it is clear that

$$Q(|f - Q(f)|) \le 2\beta^n \,\overline{Q}(|f - \overline{Q}(f)|). \tag{6.9}$$

Now, for every Whitney cube Q_j, we see from (6.6) that $\overline{Q}_j$ meets F, on which the oscillation is small. Hence $\overline{Q}_j(|f - \overline{Q}_j(f)|) \le (f^{\#})^*(t)$, for every j. Combining this with (6.9), and using the properties (6.3) and (6.4) of the Whitney decomposition, we obtain the following estimate for the L^1-norm of g:

$$\|g\|_{L^1} = \sum_j |Q_j| Q_j(|f - Q_j(f)|) \le 2\beta^n \sum_j |Q_j|(f^{\#})^*(t) \le 2\beta^n t(f^{\#})^*(t) . \tag{6.10}$$

Before estimating $\|h\|_{BMO}$ we shall need some further properties of the

Whitney cubes Q_j. For a fixed index j_o, let $J_o = \{j : Q_j \cap Q_{j_o} \neq \phi\}$. Thus J_o is the set of indices corresponding to Q_{j_o} and every cube Q_j that "touches" Q_{j_o}. The following estimate, which is a consequence of (6.5) and the dyadic nature of the cubes, shows that all of the cubes Q_j, $j \in J_o$, have approximately the same size:

$$\frac{1}{4}\operatorname{diam}(Q_{j_o}) \leq \operatorname{diam}(Q_j) \leq 4\operatorname{diam}(Q_{j_o}), \qquad j \in J_o. \tag{6.11}$$

In particular,

$$\frac{3}{2}\cdot Q_{j_o} \subseteq \bigcup_{j\in J_o} Q_j \subseteq 9.Q_{j_o} \tag{6.12}$$

The required estimate for h in (6.2) is that $\|h\|_{BMO} \leq c\,(f^{\#})^*(t)$. For this, it will suffice to show that there is a constant c, depending only on the dimension n, such that

$$Q(|h - Q(h)|) \leq c(f^{\#})^*(t), \tag{6.13}$$

for all cubes Q in R^n. In fact, if we can find any constant $\alpha = \alpha_Q$ for which

$$A(Q) \equiv Q(|h - \alpha|) \leq c(f^{\#})^*(t),$$

then (6.13) will follow, by way of (4.5).

Now fix Q and let $K = \{k : Q_k \cap Q \neq \phi\}$. On each Q_k, the function h is constant and equal to $Q_k(f)$. Hence

$$A(Q) = \sum_{k\epsilon k} \frac{|Q_k \cap Q|}{|Q|\;|Q_k|}\left|\int_{Q_k}(f(x) - \alpha)dx\right| + \frac{1}{|Q|}\int_{Q\cap F}|f(x) - \alpha|\,dx. \tag{6.14}$$

There are three cases to consider in estimating $A(Q)$:

Case 1. Suppose $K = \phi$. Then $Q \subseteq F$ and we select $\alpha = Q(f)$. Hence

$$A(Q) \leq Q(|f - Q(f)|) \leq (f^{\#})^*(t),$$

since Q contains a point of F.

Case 2. Suppose, for some $j_o \in K$, that

$$\operatorname{diam}(Q) < \frac{1}{4}\operatorname{diam}(Q_{j_o}). \tag{6.15}$$

This, and the fact that $Q_{j_o} \cap Q \neq \phi$, implies that $Q \subseteq (3/2)\cdot Q_{j_o}$ and hence, by (6.12), that $Q \cap F = \phi$. We claim that $K \subseteq J_o$, that is, any Whitney cube Q_k touching Q must also touch Q_{j_o}. The point is that, by (6.15), Q is in the interior of $(3/2)\cdot Q_{j_o}$ at a positive distance from the boundary. Thus any Q_k, $k \in K$, since it touches Q, will intersect $(3/2)\cdot Q_{j_o}$ in a set of positive measure. The first relation in (6.12) therefore shows that Q_k intersects some

Q_j, $j\varepsilon J_o$, in a set of positive measure. But the Whitney cubes have mutually disjoint interiors, so this implies $Q_k = Q_j$, that is, $k\varepsilon J_o$. Hence $K \subseteq J_o$.

Let $\tilde{Q}$ denote the cube $(\beta\cdot Q_{j_o}) = (10n^{1/2}\cdot Q_{j_o})$, and select $\alpha = \tilde{Q}(f)$. The second term on the right of (6.14) vanishes because $Q\cap F = \phi$. In the first term we use the trivial estimate $|Q_k \cap Q| \leq |Q|$, replace the index set K by the larger J_o, and use the first part of (6.11) to get

$$A(Q) \leq \sum_{j\varepsilon J_o} \frac{1}{|Q_j|} \int_{Q_j} |f(x) - \tilde{Q}(f)|dx \leq \frac{4^n}{|Q_{j_o}|} \sum_{j\varepsilon J_o} \int_{Q_j} |f(x) - \tilde{Q}(f)|dx.$$

Now $|Q_{j_o}| = \beta^{-n}|\tilde{Q}|$ and the second relation in (6.12) shows that the cubes Q_j, $j\varepsilon J_o$, are (disjoint) subsets of $\tilde{Q}$. Hence

$$A(Q) \leq (4\beta)^n \tilde{Q}(|f - \tilde{Q}(f)|) \leq (4\beta)^n (f^{\#})^*(t),$$

since $\tilde{Q}$ contains a point of F.

Case 3. Here $K \neq \phi$ and for all $k\varepsilon K$,

$$\frac{1}{4} \text{diam}(Q_k) \leq \text{diam}(Q). \tag{6.16}$$

An immediate consequence is that each Q_k, $k\varepsilon K$, is contained in $9\cdot Q$:

$$\bigcup_{k\varepsilon K} Q_k \subseteq 9\cdot Q. \tag{6.17}$$

Hence, by (6.6), the cube $\tilde{Q} = (9\beta)\cdot Q$ meets F : $\tilde{Q} \cap F \neq \phi$. Furthermore, by (6.17), the sets Q_k, $k\varepsilon K$, and $Q\cap F$ are disjoint subsets of $\tilde{Q}$. Returning to (6.14) we thus have

$$\begin{aligned} A(Q) &\leq \frac{1}{|Q|} \{\sum_{k\varepsilon K} \int_{Q_k} |f(x) - \alpha|dx + \int_{Q\cap F}|f(x) - \alpha|\} \\ &\leq \frac{1}{|Q|} \int_{\tilde{Q}} |f(x) - \alpha|dx = \frac{(9\beta)^n}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x) - \alpha|dx. \end{aligned}$$

Choosing $\alpha = \tilde{Q}(f)$, we obtain finally

$$A(Q) \leq (9\beta)^n \tilde{Q}(|f - \tilde{Q}(f)|) \leq (9\beta)^n (f^{\#})^*(t),$$

since $\tilde{Q}$ contains a point of F.

Collecting the results from all three cases, we see that $A(Q) \leq (9\beta)^n (f^{\#})^*(t)$, for any cube $Q \subseteq R^n$, and so

$$\|h\|_{BMO} \leq 2(9\beta)^n (f^{\#})^*(t).$$

This, together with (6.10), establishes the second inequality in (6.1) with $c_2 = 2\beta^n + 2(9\beta)^n$. Since $\beta = 10n^{1/2}$ depends only on n, the proof is complete.

REMARK 6.2. The key element of the proof of Theorem 6.1 is the construction of a conditional expectation of f (namely the function h) which lies in BMO. The Whitney covering, in which the cubes are arranged in "geometric progression", seems to be essential here. Since arbitrary conditional expectations do not preserve BMO-functions, this construction may be of independent interest.

REMARK 6.3. With only slight modification to the proof of Theorem 6.1, it is possible to show that the K-functional for the pair (L^p, BMO), where $0 < p < 1$, is equivalent to the functional $t(f^p_{R^n})^*(t^p)$, where $f^p_{R^n}$ is the L^p-analogue of the sharp-function defined by (4.27).

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