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INTERPOLATION OF OPERATORS FOR Λ SPACES

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Lorentz and Shimogaki [2] have characterized those pairs of Lorentz Λ spaces which satisfy the interpolation property with respect to two other pairs of Λ spaces. Their proof is long and technical and does not easily admit to generalization. In this paper we present a short proof of this result whose spirit may be traced to Lemma 4.3 of [4] or perhaps more accurately to the theorem of Marcinkiewicz [5, p. 112]. The proof involves only elementary properties of these spaces and does allow for generalization to interpolation for n pairs and for M spaces, but these topics will be reported on elsewhere.

The Banach space Λ_ϕ [1, p. 65] is the space of all Lebesgue measurable functions f on the interval $(0, l)$ for which the norm

$$\|f\|_\phi = \int_0^l f^*(s)\phi(s)\, ds$$

is finite, where ϕ is an integrable, positive, decreasing function on $(0, l)$ and f^* (the decreasing rearrangement of $|f|$) is the almost-everywhere unique, positive, decreasing function which is equimeasurable with $|f|$.

A pair of spaces $(\Lambda_\phi, \Lambda_\psi)$ is called an interpolation pair for the two pairs $(\Lambda_{\phi_1}, \Lambda_{\psi_1})$ and $(\Lambda_{\phi_2}, \Lambda_{\psi_2})$ if each linear operator which is bounded from Λ_{ϕ_i} to Λ_{ψ_i} (both $i=1, 2$) has a unique extension to a bounded operator from Λ_ϕ to Λ_ψ.

THEOREM (LORENTZ-SHIMOGAKI). *A necessary and sufficient condition that $(\Lambda_\phi, \Lambda_\psi)$ be an interpolation pair for $(\Lambda_{\phi_1}, \Lambda_{\psi_1})$ and $(\Lambda_{\phi_2}, \Lambda_{\psi_2})$ is that there exist a constant A independent of s and t so that*

$$(*) \qquad \Psi(t)/\Phi(s) \leqq A \max_{i=1,2}(\Psi_i(t)/\Phi_i(s))$$

holds, where $\Phi(s)=\int_0^s \phi(r)\, dr, \cdots, \psi_2(t)=\int_0^t \Psi_2(r)\, dr$.

PROOF. We only sketch the proof of the necessity since it is standard.

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Suppose there are numbers s_n and t_n in $(0, l)$ such that $\Psi(t_n)/\Phi(s_n) > n^3 \max_{i=1,2}(\Psi_i(t_n)/\Phi_i(s_n))$. Define the positive operator

$$T_n f(t) = \left(C_n \int_0^{s_n} f(s)\, ds/s_n\right)\chi_{(0,t_n)}(t),$$

where $C_n = \min_{i=1,2}(\Phi_i(s_n)/\Psi_i(t_n))$.

For each f in Λ_{ϕ_i}, $T_n f$ belongs to Λ_{ψ_i} and T_n has operator norm less than or equal to 1, but as an operator from Λ_ϕ to Λ_ψ, T_n has operator norm larger than n^3. Hence the operator $T = \sum_1^\infty T_n/n^2$ is a bounded operator from Λ_{ϕ_i} to Λ_{ψ_i} $(i=1, 2)$, but T is not a bounded operator from Λ_ϕ to Λ_ψ.

To show that condition $(*)$ is sufficient, we prove that

$$(1) \qquad \|Tf\|_\psi \leqq 2AM\,\|f\|_\phi$$

where M is the maximum of the operator norms of T acting from Λ_ϕ to Λ_{ψ_i} $(i=1, 2)$. We can assume that f is an arbitrary simple function with finite support since these functions are dense in Λ_ϕ. We can also require f to be positive since $\|f\|_\phi = \|\,|f|\,\|_\phi$. Each function of this type can be written as $f = \sum_1^n \alpha_i \chi_{E_i}$ where the α_i's are positive and $E_n \subset \cdots \subset E_1$. Hence $f^* = \sum_1^n \alpha_i \chi_{(0,a_i)}$ where $a_i = mE_i$. But then

$$(2) \qquad \|T\chi_E\|_\psi \leqq 2AM\Phi(mE), \quad \text{all measurable } E \subset (0, l)$$

is equivalent to relation (1), since

$$\|Tf\|_\psi \leqq \sum_1^n \alpha_i\,\|T\chi_E\|_\psi \leqq 2AM \sum_1^n \alpha_i \Phi(a_i) = 2AM\,\|f\|_\phi\,.$$

Hence, if we let $g = (T\chi_E)^*$, the proof is reduced to the following

LEMMA. *Suppose condition* $(*)$ *holds and* g *is a positive decreasing function that satisfies*

$$(3) \qquad \|g\|_{\psi_i} \leqq M\Phi_i(a) \qquad (i = 1, 2),$$

then

$$(4) \qquad \|g\|_\psi \leqq 2AM\Phi(a).$$

PROOF. First assume g is a step function with finite support, i.e., $g = \sum_1^m \beta_j \chi_{(0,t_j)}$. Set $J = \{j \mid \max_{i=1,2}(\Psi_i(t_j)/\Phi_i(a)) = \Psi_1(t_j)/\Phi_1(a)\}$ and then let $g_1 = \sum_{j\in J} \beta_j \chi_{(0,t_j)}$ and $g_2 = g - g_1$. Notice that both functions are positive, decreasing, step functions and

$$(5) \qquad \|g_i\| \leqq \|g\|$$

in any Λ space. Now using condition $(*)$, relations (5) and (3), we have

$$\begin{aligned}\|g_1\|_\psi/\Phi(a) &= \sum_{j\in J} \beta_j \Psi(t_j)/\Phi(a)\\ &\leqq A \sum_{j\in J} \beta_j \max_{i=1,2}(\Psi_i(t_j)/\Phi_i(a))\\ &= A \sum_{j\in J} \beta_j \Psi_1(t_j)/\Phi_1(a) = A\ \|g_1\|_{\psi_1}/\Phi_1(a)\\ &\leqq A\ \|g\|_{\psi_1}/\Phi_1(a) \leqq AM.\end{aligned}$$

Similarly

$$\|g_2\|_\psi/\Phi(a) \leqq A\ \|g_2\|_{\psi_2}/\Phi_2(a) \leqq AM.$$

Hence, we obtain relation (4) for positive, decreasing, step functions.

Now suppose g is an arbitrary positive decreasing function and let $\{g_n\}$ be a monotone increasing sequence of positive decreasing step functions converging pointwise to g. By (3) and (5)

$$\|g_n\|_{\psi_i} \leqq M\Phi_i(a) \qquad (i = 1, 2)$$

so

$$\|g_n\|_\psi \leqq 2AM\Phi(a).$$

Applying the monotone convergence theorem to $\{g_n\psi\}$, we obtain relation (4).

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