Interpolation Theorems for Compact Operators

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Communicated by the Editors

§1. **Introduction.** In [3] and [7] interpolation theorems of the following kind were considered:

Each linear operator which is a bounded linear operator from X_i to Y_i, $i = 1, 2$, and which is also a compact operator in one of the two cases has a unique extension to a compact operator from X to Y.

Such pairs (X, Y) are said to be compact intermediate for the interpolation segment $\sigma = [(X_1, Y_1), (X_2, Y_2)]$. All spaces in [3] were L^p spaces, while in [7], $X_1 = Y_1 = L^p$, $X_2 = Y_2 = L^\infty$ (or dually, $X_1 = Y_1 = L^1$, $X_2 = Y_2 = L^q$), and $X = Y$ were rearrangement invariant spaces. We give extensions and generalizations of theorems of this kind using Calderón's operator S_σ in Sections 3 and 4 and the compression (or dilation) operator σ_a in Sections 5 and 6.

In Section 2 we define rearrangement invariant Banach function spaces, the fundamental function of such a space, and Calderón's operator S_σ and give their various properties. Several theorems from [11] are stated for completeness.

A sufficient condition is given in Section 3 for weak compact (and hence compact) interpolation in terms of a function F which depends in a simple manner on the fundamental functions of the six spaces involved. This condition is shown to be necessary if the intermediate spaces are extremal. In Section 4 we obtain necessary and sufficient conditions for the scale $\Lambda_\alpha(X)$, $0 < \alpha \leqq 1$, to be weak compact intermediate for an interpolation segment.

In Section 5 we get a sufficient condition for compact interpolation involving the Boyd indices of the six spaces when the underlying measure space is infinite or when the slope of the interpolation segment is positive. This condition is necessary when $X = Y$ and $\sigma = [(L^p, L^p), (L^q, L^q)]$. Section 6 deals with the remaining case, *i.e.*, finite measure space and an interpolation segment with negative slope. This requires use of an "adjoint" operator and a duality argument.

The results of §3 and §4 deal with weak compact interpolation and do not give Krasnosel"skii's theorem, whereas Theorems 3 and 4 of §5 and §6, respectively, do. Also Theorems 1 and 2 do not follow from the results of §5 and §6 since the operator S_σ does not belong to the class of compact operators in

Indiana University Mathematics Journal, ©, Vol. 22, No. 10 (1973)

consideration. However, Lemma 1 is common and essential to both types of theorems.

§2. Rearrangement invariant spaces and interpolation. We consider Banach spaces X of real valued Lebesgue measurable functions on a possibly infinite interval $(0, \ell)$ which satisfy

(2.1) $0 \leqq |g| \leqq |f|$ a.e., and $f \in X$, then $g \in X$ and $\|g\|_X \leqq \|f\|_X$.

(2.2) $0 \leqq f_n \nearrow f$ a.e., $f_n \in X$, and $\|f_n\|_X \leqq M$, then $f \in X$ and $\|f\|_X \leqq M$.

(2.3) $mE < \infty$ implies the existence of a constant $C_E > 0$ such that $\int_E f\, ds \leqq C_E \|f\|_X$ for each $f \in X$.

(2.4) $mE < \infty$ implies $\|\chi_E\|_X < \infty$ where χ_E is the characteristic function of the set E.

Spaces of this type are usually referred to as Banach function spaces (see [5], [8]). Notice that property (2.3) gives that X is continuously embedded in the topological vector space $\mathfrak{M}$ of measurable functions on $(0, \ell)$ with the topology of convergence in measure on subsets of finite measure. A Banach function space X is called a rearrangement invariant space if $m\{s \mid |g(s)| > \epsilon\}$ is finite for each positive ϵ and each $g \in X$, and the following condition is satisfied:

If $g \in X$ and g' is any function on $(0, \ell)$ equimeasurable with g, then $g' \in X$ and $\|g'\|_X = \|g\|_X$.

The decreasing rearrangement f^* of a function $f \in X$ is the almost everywhere unique positive decreasing function on $(0, \ell)$ which is equimeasurable with $|f|$. So, for a rearrangement invariant space we have

$$\|f\|_X = \|f^*\|_X .$$

We note here that one could replace in our theorems the interval $(0, \ell)$ by any nonatomic σ-finite measure space (see, for example, [3]).

The associate space X' of a rearrangement invariant space X is defined by

$$X' = \left\{ g \mid \|g\|_{X'} = \sup_{\|f\|_X \leq 1} \int_0^\ell |fg|\, dt < \infty \right\}.$$

It follows that X' is also a rearrangement invariant space under the norm $\|\cdot\|_{X'}$. A result due to Lorentz and Luxemburg [8] is that X is isometrically isomorphic to X'', and in particular

$$\|f\|_X = \sup_{\|g\|_{X'} \leq 1} \int_0^\ell f^*(s) g^*(s)\, ds.$$

The fundamental function, φ_X, of a rearrangement invariant space X is given by

$$\varphi_X(t) = \|\chi_{(0,t)}\|_X , \qquad t \in (0, \ell).$$

The fundamental function is a positive nondecreasing function on $(0, \ell)$ which is absolutely continuous and satisfies

$$\varphi_X(t)\varphi_{X'}(t) = t, \qquad \text{for all } t.$$

From this it follows that

$$\frac{d\varphi_X(t)}{dt} \leqq \frac{\varphi_X(t)}{t}, \quad \text{a.e.} \tag{2.5}$$

In [14] and [15] it was shown that X can be equivalently renormed so that the resulting fundamental function is concave, and hence its derivative is a positive, nonincreasing function. We assume without loss of generality throughout this paper that all spaces are renormed in this manner. The rearrangement invariant Lorentz spaces associated with X, defined by

$$\Lambda(X) = \left\{f \mid \|f\|_{\Lambda(X)} = \int_0^t f^*(s)\, d\varphi_X(s) < \infty\right\}$$

and

$$M(X) = \left\{f \mid \|f\|_{M(X)} = \sup_t \left(\int_0^t f^*(s)\, ds\, \varphi_X(t)/t\right) < \infty\right\},$$

are extremal in the sense that

$$\Lambda(X) \subset X \subset M(X) \tag{2.6}$$

with normal continuous embeddings (see relation (2.1) of [11]). The Lorentz Λ and M spaces satisfy the following dualities

$$\Lambda(X)^* = \Lambda(X)' = M(X') \tag{2.7}$$

and

$$M^\circ(X)^* = M^\circ(X)' = \Lambda(X')$$

where X^* is the Banach space dual of X and $M^\circ(X)$ is the norm closure of the simple functions with finite support in $M(X)$ (see [5], [8]).

The following two classes of spaces will be useful in Sections 3 and 4:

(2.8) $X \in \mathfrak{U}$ if for some $\alpha < 1$ there is a pair of positive constants C and δ such that

$$\varphi_X(u)/\varphi_X(v) \leqq C(u/v)^\alpha \quad \text{if} \quad v/u < \delta.$$

(2.9) $X \in \mathfrak{L}$ if for some $\beta > 0$, there is a pair of positive constants C and δ such that

$$\varphi_X(v)/\varphi_X(u) \leqq C(v/u)^\beta \quad \text{if} \quad v/u < \delta.$$

The fundamental indices of a rearrangement invariant space X which were introduced by Zippin [15] are given by

$$\underline{\gamma}_X = \lim_{t\to 0} \theta_1(t) = \sup_{0<t<1} \theta_1(t),$$

$$\bar{\gamma}_X = \lim_{t\to\infty} \theta_1(t) = \inf_{t>1} \theta_1(t),$$

where $\theta_1(t) = \log M(t, X)/\log t$ and $M(t, X) = \sup_s (\varphi_X(st)/\varphi_X(s))$. If $m(t, X) =$

$\inf_s (\varphi_X(st)/\varphi_X(s))$, then we have for each $\epsilon > 0$ the existence of a positive δ such that

$$M(t, X) \leqq t^{\underline{\gamma}_X - \epsilon}, \qquad 0 < t < \delta, \tag{2.10}$$

$$M(t, X) \leqq t^{\bar{\gamma}_X + \epsilon}, \qquad 1/\delta < t, \tag{2.11}$$

$$t^{\bar{\gamma}_X + \epsilon} \leqq m(t, X), \qquad 0 < t < \delta, \tag{2.12}$$

$$t^{\underline{\gamma}_X - \epsilon} \leqq m(t, X), \qquad 1/\delta < t. \tag{2.13}$$

Hence $X \in \mathfrak{U}$ (resp. $X \in \mathfrak{L}$) is equivalent to $\bar{\gamma}_X < 1$ (resp. $\underline{\gamma}_X > 0$).

An operator T is of weak type (X, Y) if it satisfies

$$\sup_t ((Tf)^*(t)\varphi_Y(t)) \leqq A \, \|f\|_{\Lambda(X)} \tag{2.14}$$

where A is independent of $f \in \Lambda(X)$. The set of all such operators will be denoted by $W(X, Y)$, and the smallest constant A in (2.14) will be written $\|T\|_{W(X,Y)}$. Let $B(X, Y)$ denote the bounded linear operators from X to Y. In [11] it was shown that $W(X, Y) = B(\Lambda(X), M(Y))$ if $Y \in \mathfrak{U}$. The operators which are simultaneously of weak types (X_i, Y_i), $i = 1, 2$, will be denoted by $W(\sigma)$, where $\sigma = [(X_1, Y_1), (X_2, Y_2)]$. A pair of rearrangement invariant spaces (X, Y) is said to be weak intermediate for an interpolation segment σ if each operator belonging to $W(\sigma)$ has a unique extension belonging to $B(X, Y)$.

The ideal of linear operators which map norm bounded subsets of X into sequentially precompact subsets of Y (*i.e.*, compact operators from X to Y) is denoted $K(X, Y)$. A pair of rearrangement invariant spaces (X, Y) will be called weak compact intermediate for σ if each operator which belongs to $W(\sigma) \cap (K(\Lambda(X_1), \mathfrak{M}) \cup K(\Lambda(X_2), \mathfrak{M}))$ has a unique extension belonging to $K(X, Y)$. It is obvious that each pair which is weak compact intermediate is also compact intermediate for σ by the continuous embeddings in relation (2.6). A pair (X, Y) is simply called intermediate for σ if $B(X_1, Y_1) \cap B(X_2, Y_2)$ is continuously embedded in $B(X, Y)$. This is known (see [10]) to be equivalent to the property that each operator belonging to the intersection has a unique extension belonging to $B(X, Y)$.

For a given interpolation segment $\sigma = [(X_1, Y_1), (X_2, Y_2)]$ and a pair (X, Y) define

$$\Psi(s, t) = \min_{i=1,2} (\varphi_{X_i}(s)/\varphi_{Y_i}(t)), \qquad s, t \in (0, \ell),$$

and

$$F(s, t) = \Psi(s, t)\varphi_Y(t)/\varphi_X(s), \qquad s, t \in (0, \ell).$$

The function $\Psi(s, t)$ is a concave, absolutely continuous function of s, and we denote by $\Phi(s, t)$ its positive decreasing derivative. The Calderón operator S_σ is defined pointwise by

$$S_\sigma(f)(t) = \int_0^\ell f(s)\Phi(s, t) \, ds$$

when this integral is absolutely convergent. Since $\Phi(\cdot, t)$ decreases, we have

$$|S_\sigma(f)(t)| \leqq S_\sigma(f^*)(t). \tag{2.15}$$

We include several results from [11] which will be used in the next two sections.

Proposition 1 *(Lemma (4.2), Theorem (4.5)). The operator S_σ belongs to $W(\sigma)$. Moreover, each operator which belongs to $W(\sigma)$ must satisfy*

$$(Tf)^*(t) \leqq 2(\max_{i=1,2} ||T||_{W(X_i, Y_i)}) S_\sigma(f^*)(t)$$

for all t and all f belonging to $\Lambda(X_1) + \Lambda(X_2)$, where the sum of two spaces $X + Y = \{f \mid ||f||_{X+Y} = \inf_{f_1+f_2=f} (||f_1||_X + ||f_2||_Y) < \infty\}$.

Proposition 2 *(Theorem (4.7), Corollary (4.4)). A necessary and sufficient condition that (X, Y) be weak intermediate for σ is that S_σ belong to $B(X, Y)$. If this is the case, then X is continuously embedded in $\Lambda(X_1) + \Lambda(X_2)$.*

Proposition 3 *(Lemma (6.8)). The space $\Lambda(X_1) + \Lambda(X_2)$ is identical with $\Lambda_\phi = \{f \mid ||f||_{\Lambda\phi} = \int_0^\ell f^*(s)\, d\Phi(s) < \infty\}$ where $\Phi(s) = \min (\varphi_{X_1}(s), \varphi_{X_2}(s))$ and ϕ is the derivative of Φ.*

Proposition 4 *(Theorem (5.1)). Suppose Y belongs to the class $\mathfrak{L}$. A necessary and sufficient condition that $(\Lambda(X), \Lambda(Y))$ be weak intermediate for σ is that*

$$\sup_s \left(\int_0^\ell F(s, t)\, dt/t\right) < \infty.$$

Proposition 5 *(Theorem (5.2)). Suppose X belongs to $\mathfrak{L}$ and Y belongs to $\mathfrak{U}$. A necessary and sufficient condition that $(M(X), M(Y))$ be weak intermediate for σ is that*

$$\sup_t \left(\int_0^\ell F(s, t)\, ds/s\right) < \infty.$$

Proposition 6 *(Theorem (5.9)). A sufficient condition for $(M(X), \Lambda(Y))$ to be weak intermediate for σ is that*

$$\int_0^\ell \int_0^\ell F(s, t)\, ds/s\, dt/t < \infty.$$

If X_1, X_2, and Y belongs to $\mathfrak{L}$ and X belongs to $\mathfrak{U}$, then the condition is necessary.

§3. A sufficient condition. A function f in a rearrangement invariant space X is said to have absolutely continuous norm if $||f\chi_E||_X \to 0$ as $E \searrow \phi$. The norm is called absolutely continuous if each function in X has absolutely continuous norm. A subset V of X is uniformly absolutely continuous if $||f\chi_E||_X$ goes to zero uniformly in $f \in V$ with $E \searrow \phi$. It is easy to see that a necessary and sufficient condition for a subset V, each function of which has absolutely continuous norm, to be sequentially precompact is that V be uniformly absolutely continuous and be sequentially precompact in $\mathfrak{M}$.

The following lemma is basic for our development.

Lemma 1. *Suppose that for some* t_0

$$\int_0^t F(s, t_0)\, ds/s < \infty \tag{3.1}$$

and that T *belongs to* $W(\sigma) \cap (K(\Lambda(X_1), \mathfrak{M}) \cup K(\Lambda(X_2), \mathfrak{M}))$, *then* T *belongs to* $K(X, \mathfrak{M})$.

Proof. It suffices to show that $V = \{Tf \mid \|f\|_X \leqq 1\}$ is sequentially precompact in $\mathfrak{M}$. Suppose $\|f\|_X \leqq 1$. Then define

$$f^{(k)}(s) = \chi_{(0,k)}(s) \min(|f(s)|, k) \operatorname{sgn} f(s).$$

Since f^* is a positive decreasing function, relation (2.6) gives

$$f^*(s)\varphi_X(s) \leqq \int_0^s f^*(t)\, dt\, \varphi_X(s)/s \leqq \|f\|_{M(X)} \leqq \|f\|_X \leqq 1.$$

Hence

$$f^*(s) \leqq 1/\varphi_X(s), \qquad \text{all } s. \tag{3.2}$$

We show that $f^{(k)}$ converges uniformly in the norm of the space $\Lambda(X_1) + \Lambda(X_2)$ to f for all $f \in V$. In order to do this, we first show that $1/\varphi_X$ belongs to $\Lambda(X_1) + \Lambda(X_2)$,

$$\begin{aligned} \|1/\varphi_X\|_{\Lambda(X_1)+\Lambda(X_2)} &= \int_0^t 1/\varphi_X(s)\, d\min(\varphi_{X_1}(s), \varphi_{X_2}(s)) \\ &\leqq \frac{\max_{i=1,2} \varphi_{Y_i}(t_0)}{\varphi_Y(t_0)} \int_0^t \frac{\varphi_Y(t_0)}{\varphi_X(s)} \Phi(s, t_0)\, ds \\ &\leqq C \int_0^t F(s, t_0)\, ds/s < \infty \end{aligned}$$

since $\Phi(s, t_0) \leqq \Psi(s, t_0)/s$ by (2.5). By Proposition 3, $\Lambda(X_1) + \Lambda(X_2)$ has absolutely continuous norm. Hence, by (3.2) we have

$$\begin{aligned} \|f - f^{(k)}\|_{\Lambda(X_1)+\Lambda(X_2)} &= \|(f^* - k)_+\| \leqq \|(1/\varphi_X - k)_+\| \\ &= \|1/\varphi_X - (1/\varphi_X)^{(k)}\|. \end{aligned} \tag{3.3}$$

But $(1/\varphi_X)^{(k)} \nearrow 1/\varphi_X$, so for $\epsilon > 0$, there is an N so that the right hand side of (3.3) is less than ϵ if $k \geqq N$. So the convergence of $f^{(k)}$ to f, $\|f\|_X \leqq 1$, is uniform in $\Lambda(X_1) + \Lambda(X_2)$.

Suppose now that T belongs to $W(\sigma) \cap K(\Lambda(X_1), \mathfrak{M})$, and let $\{f_n\}$ be a sequence from the unit ball of X. The sequence $\{f_n^{(1)}\}$ is norm bounded in $\Lambda(X_1)$,

$$\|f_n^{(1)}\|_{\Lambda(X_1)} \leqq \varphi_{X_1}(1), \qquad \text{all } n,$$

so there is a subsequence $\{f_{1,n}\}$ of $\{f_n\}$ such that $\{T(f_{1,n}^{(1)})\}$ converges in $\mathfrak{M}$ to some h^1. The sequence $\{f_{1,n}^{(2)}\}$ is norm bounded in $\Lambda(X_1)$, so there is a subsequence $\{f_{2,n}\}$ of $\{f_{1,n}\}$ such that $\{T(f_{2,n}^{(2)})\}$ converges in $\mathfrak{M}$ to some h^2. At stage m, $\{f_{m,n}^{(m+1)}\}_{n=1}^{\infty}$ is norm bounded in $\Lambda(X_1)$; in fact,

$$||f_{m,n}^{(m+1)}||_{\Lambda(X_1)} \leqq (m+1)\varphi_{X_1}(m+1).$$

Hence we can find a subsequence $\{f_{m+1,n}\}_{n=1}^{\infty}$ of $\{f_{m,n}\}_{n=1}^{\infty}$ such that $\{T(f_{m+1,n}^{(m+1)})\}$ converges in $\mathfrak{M}$ to some h^{m+1}.

The diagonal sequence $g_n = f_{n,n}$ is a subsequence of f_n, and we show that Tg_n is Cauchy in $\mathfrak{M}$. Since $T \in W(\sigma)$, T is a continuous linear operator from $\Lambda(X_1) + \Lambda(X_2)$ into $\mathfrak{M}$. But $f^{(k)}$ converges to f in $\Lambda(X_1) + \Lambda(X_2)$ uniformly in $||f||_X \leqq 1$, so since

$$Tg_n - Tg_m = T(g_n - g_n^{(k)}) + T(g_m^{(k)} - g_m) + T(g_n^{(k)} - g_m^{(k)})$$

and $\{g_n\}_{n\geqq k}$ is a subsequence of $\{f_{k,n}\}_{n=1}^{\infty}$, Tg_n is therefore Cauchy in measure on subsets of $(0, \ell)$ of finite measure.

Lemma 2. *Suppose Y is isometrically isomorphic to X^*. If the function $\Phi(\cdot, t)$ belongs to X for each t, then S_σ belongs to $K(Y, \mathfrak{M})$.*

Proof. Consider the natural (Y, X) pairing given by

$$\langle g, h\rangle = \int_0^{\ell} g(s)h(s)\,ds, \qquad g \in Y, \quad h \in X.$$

The $\sigma(Y, X)$ topology is the weak* topology, so the Bourbaki-Alaoglu theorem produces for each sequence $\{f_n\}$ in the unit ball of Y a subsequence $\{g_n\}$ and a function g_0 such that

$$\langle g_n, h\rangle \to \langle g_0, h\rangle \quad \text{for each} \quad h \in X.$$

But $\Phi(\cdot, t) \in X$ for all t, so

$$S_\sigma(g_n)(t) \to S_\sigma(g_0)(t) \quad \text{for all } t.$$

In particular, $S_\sigma(g_n)$ converges to $S_\sigma(g_0)$ in $\mathfrak{M}$.

Lemma 3. *Suppose $\ell < \infty$ and*

$$\lim_{s\to 0} \varphi_{X_2}(s)/\varphi_{X_1}(s) = 0 \tag{3.4}$$

holds, then S_σ belongs to $W(\sigma) \cap K(\Lambda(X_1), \mathfrak{M})$.

Proof. By Proposition 1, Lemma 2, and Property (2.7), it suffices to show that $\Phi(\cdot, t)$ belongs to $M^\circ(X_1')$. Since

$$\begin{aligned}||\Phi(\cdot, t)||_{M(X_1')} &= \sup_s \left(\int_0^s \Phi(r, t)\,dr\, \varphi_{X_1'}(s)/s\right)\\ &= \sup_s (\Psi(s, t)/\varphi_{X_1}(s)) \leqq 1/\varphi_{Y_1}(t),\end{aligned}$$

we need only show that $\lim_{s\to 0} ||\Phi(\cdot, t)\chi_{(0,s)}||_{M(X_1')} = 0$. But

$$\lim_{s\to 0} \|\Phi(\cdot, t)\chi_{(0,s)}\|_{M(X_1')} = \lim_{s\to 0}\left(\sup_{r\leqq s} \frac{\Psi(r, t)}{\varphi_{X_1}(r)}\right) = \lim_{s\to 0} \frac{\varphi_{X_2}(s)}{\varphi_{X_1}(s)} \frac{1}{\varphi_{Y_2}(t)} = 0.$$

An obvious condition that implies (3.4) holds is that $\bar{\gamma}_{X_1} < \underline{\gamma}_{X_2}$. This is the case for $X_1 = L^{pr}$, $X_2 = L^{qr'}$, $p > q$, $1 \leqq r, r' \leqq \infty$.

Lemma 4. A sufficient condition for S_σ to belong to $K(M(X), \Lambda(Y))$ is that

$$\int_0^\ell \int_0^\ell F(s, t)\, ds/s\, dt/t < \infty. \tag{3.5}$$

In this case, $V = \{S_\sigma(f) \mid \|f\|_{M(X)} \leqq 1\}$ is uniformly absolutely continuous in $\Lambda(Y)$.

Proof. We show that S_σ belongs to $K(M(X), \mathfrak{M})$ and that V is uniformly absolutely continuous in $\Lambda(Y)$. Since $\Lambda(X')^* = M(X)$, the first assertion is proved by an application of Lemma 2 to $M(X)$ and $\Lambda(X')$, *i.e.*, we show $\Phi(\cdot, t) \in \Lambda(X')$ for each t. By Proposition 6 we know that $S_\sigma(f) \in \Lambda(Y)$ for each $f \in M(X)$. In particular,

$$\left|\int_0^\ell f(s)\Phi(s, t)\, ds\right| \leqq S_\sigma(f^*)(t) < \infty, \qquad f \in M(X).$$

But this gives that $\Phi(\cdot, t)$ belongs to $M(X)' = \Lambda(X')$ (see Lemma 4, page 7 of [8]).

To show that V is uniformly absolutely continuous in $\Lambda(Y)$, suppose $\|f\|_X \leqq 1$ and that (3.5) holds. Then $S_\sigma(1/\varphi_X) \in \Lambda(Y)$ and relation (3.2) implies

$$\|S_\sigma(f)\chi_E\|_{\Lambda(Y)} \leqq \|S_\sigma(f^*)\chi_E\| \leqq \|S_\sigma(1/\varphi_X)\chi_E\|.$$

But the right hand side tends to zero as $E \searrow \phi$ since $\Lambda(Y)$ has absolutely continuous norm.

Theorem 1. Suppose ℓ is finite. Then a sufficient condition that (X, Y) be weak compact (and hence compact) intermediate for σ is that condition (3.5) hold. If (3.4) holds, and all spaces belong to the class $\mathfrak{L} \cap \mathfrak{U}$, then condition (3.5) is necessary for $X = M(X)$ and $Y = \Lambda(Y)$.

Proof. Suppose T belongs to $W(\sigma) \cap K(\Lambda(X_1), \mathfrak{M})$. Condition (3.5) implies that (3.1) is true, so Lemma 1 shows that $T \in K(X, \mathfrak{M})$. We need only show that $\{Tf \mid \|f\|_X \leqq 1\}$ is uniformly absolutely continuous in Y. Let $mE \to 0$. Then by Proposition 1

$$\|(Tf)\chi_E\|_Y \leqq \|(Tf)^*\chi_{(0,mE)}\|_Y \leqq \text{const } \|S_\sigma(f^*)\chi_{(0,mE)}\|_Y$$
$$\leqq \text{const } \|S_\sigma(f^*)\chi_{(0,mE)}\|_{\Lambda(Y)}.$$

But the last term on the right is uniformly small for mE small by Lemma 4. Property (2.6) then gives that $T \in K(X, Y)$.

Suppose on the other hand that $(M(X), \Lambda(Y))$ is weak compact intermediate for σ and that (3.4) holds. By Lemma 3, $S_\sigma \in W(\sigma) \cap K(\Lambda(X_1), \mathfrak{M})$, so S_σ must be compact from $M(X)$ to $\Lambda(Y)$. But $K(X, Y) \subset B(X, Y)$ and Proposition 6 gives that (3.5) is true.

§4. Weak compact interpolation in $\Lambda_\alpha(X)$. In [11], compare also [6], rearrangement invariant spaces associated with X, $\Lambda_\alpha(X)$, $0 < \alpha \leqq 1$, were introduced and shown to have many of the properties of the Lorentz L^{pq} spaces. These spaces are defined by

$$\Lambda_\alpha(X) = \left\{ f \mid \|f\|^*_{\Lambda_\alpha(X)} = \left(\int_0^\ell [f^*(s)\varphi_X(s)]^{1/\alpha}\, ds/s \right)^\alpha < \infty \right\} \tag{4.1}$$

where $0 < \alpha \leqq 1$. In general, $\|\cdot\|^*$ is only a quasi-norm, but if we assume that $X \in \mathfrak{L}$, as we do throughout this section, there is an equivalent (up to constants) norm given by

$$\|f\|_{\Lambda_\alpha(X)} = \left\{ \int_0^\ell \left[\int_0^t f^*(s)\, ds\, \varphi_X(t)/t \right]^{1/\alpha} dt/t \right\}^\alpha. \tag{4.2}$$

In [11] it was shown that for $0 < \beta \leqq \alpha \leqq 1$

$$\Lambda(X) \subset \Lambda_\alpha(X) \subset \Lambda_\beta(X) \subset M^\circ(X)$$

and that after equivalent renorming and in the limit

$$\Lambda_1(X) = \Lambda(X), \qquad \Lambda_0(X) = M^\circ(X)$$

holds. Hence the spaces $\Lambda_\alpha(X)$ form a continuous scale between $\Lambda(X)$ and $M^\circ(X)$.

The proof of the following lemma is essentially due to W. Orlicz (see [5], page 31, Theorem 2.1.2). If we define the measure $\mu(E) = \int_E ds/s$, then we have

Lemma 5. *Suppose $K(s, t) \geqq 0$. Then the positive integral operator given by*

$$Tf(t) = \int_0^\ell f(s)K(s, t)\, ds$$

satisfies

$$\|T\|_{L^{1/\alpha}(\mu)} \leqq M_1^\alpha M_0^{1-\alpha}, \qquad 0 \leqq \alpha \leqq 1, \tag{4.3}$$

where

$$M_1 = \|T\|_{L^1(\mu)} = \sup_s \int_0^\ell K(s, t)\, dt/t \tag{4.4}$$

and

$$M_0 = \|T\|_{L^\infty(\mu)} = \sup_t \int_0^\ell K(s, t)\, ds/s. \tag{4.5}$$

Lemma 6. *For each $0 < \alpha \leqq 1$, $V = \{S_\sigma(f) \mid \|f\|_{\Lambda_\alpha(X)} \leqq 1\}$ is uniformly absolutely continuous in $\Lambda_\alpha(Y)$, if*

$$\sup_t \int_0^\ell F(s,t)\,ds/s = K_2 < \infty, \tag{4.6}$$

$$\sup_s \int_0^\ell F(s,t)\,dt/t = K_1 < \infty \tag{4.7}$$

and

$$\lim_{E\searrow\phi} \left(\sup_s \int_E F(s,t)\,dt/t\right) = 0. \tag{4.8}$$

Proof. For each $E \subset (0, \ell)$, define the positive integral operator $T_E(f)(t) = \int_0^\ell f(s)F(s,t)\,ds/s\ \chi_E(t)$. By Lemma 5 we have

$$\begin{aligned} \|T_E\|_{L^{1/\alpha}(\mu)} &\leqq \left(\sup_s \int_E F(s,t)\,dt/t\right)^\alpha \left(\sup_t \int_0^\ell \chi_E(t)F(s,t)\,ds/s\right)^{1-\alpha} \\ &\leqq K_2^{1-\alpha}\left(\sup_s \int_E F(s,t)\,dt/t\right)^\alpha, \end{aligned}$$

and so

$$\lim_{E\searrow\phi} \|T_E\|_{L^{1/\alpha}(\mu)} = 0. \tag{4.9}$$

By property (2.5) of fundamental functions we have

$$\begin{aligned} S_\sigma(f^*)(t)\varphi_Y(t)\chi_E(t) &= \int_0^\ell f^*(s)\varphi_Y(t)\Phi(s,t)\,ds\,\chi_E(t) \\ &\leqq \int_0^\ell f^*(s)\varphi_Y(t)\Psi(s,t)\,ds/s\,\chi_E(t) \\ &= T_E(f^*\varphi_X)(t). \end{aligned} \tag{4.10}$$

Hence for $0 < \alpha \leqq 1$, there holds by properties (2.15), (4.2), and (4.10)

$$\begin{aligned} \|S_\sigma(f)\chi_E\|_{\Lambda_\alpha(Y)} &\leqq \text{const } \|S_\sigma(f^*)\varphi_Y\chi_E\|_{L^{1/\alpha}(\mu)} \\ &\leqq \text{const } \|T_E(f^*\varphi_X)\|_{L^{1/\alpha}(\mu)} \\ &\leqq \text{const } \|T_E\|_{L^{1/\alpha}(\mu)}\ \|f^*\varphi_X\|_{L^{1/\alpha}(\mu)} \\ &\leqq \text{const } \|T_E\|_{L^{1/\alpha}(\mu)}\ \|f\|_{\Lambda_\alpha(X)}\ . \end{aligned} \tag{4.11}$$

But by (4.9) the right most expression of (4.11) becomes uniformly small for $\|f\|_{\Lambda_\alpha(X)} \leqq 1$.

Theorem 2. Suppose ℓ is finite and conditions (4.6), (4.7), *and* (4.8) *hold. Then for* $0 < \alpha \leqq 1$, $(\Lambda_\alpha(X), \Lambda_\alpha(Y))$ *is weak compact intermediate for* σ. *If condition* (3.4) *holds, and all spaces belong to* $\mathfrak{L} \cap \mathfrak{U}$, *then conditions* (4.7) *and* (4.8) *are necessary for* $(\Lambda(X), \Lambda(Y))$ *to be weak compact intermediate for* σ.

Proof. Suppose for simplicity that $T \in W(\sigma) \cap K(\Lambda(X_1), \mathfrak{M})$. We show that T is a compact operator from $\Lambda_\alpha(X)$ to $\Lambda_\alpha(Y)$, $0 < \alpha \leqq 1$. But since (4.6) trivially implies condition (3.1) is true, Lemma 1 gives that T belongs to $K(M(X), \mathfrak{M})$. Hence, in particular, T belongs to $K(\Lambda_\alpha(X), \mathfrak{M})$ and so we need only show that $\{Tf \mid \|f\|_{\Lambda_\alpha(X)} \leqq 1\}$ is uniformly absolutely continuous in $\Lambda_\alpha(Y)$. Notice that Proposition 1 implies that

$$\|\chi_E Tf\|_{\Lambda_\alpha(Y)} \leqq 2(\max_{i=1,2} \|T\|_{W(X_i,Y_i)}) \|\chi_{(0,mE)} S_\sigma(f^*)\|_{\Lambda_\alpha(Y)} .$$

But by Lemma 6 the right hand side goes to zero as $E \searrow \phi$ uniformly in $\|f\|_{\Lambda_\alpha(X)} \leqq 1$.

On the other hand, suppose $(\Lambda(X), \Lambda(Y))$ is weak compact intermediate for σ. Proposition 4 implies immediately that (4.8) holds. Condition (4.7) is exactly the condition that $S_\sigma \in B(\Lambda(X), \Lambda(Y))$. Condition (4.8) is equivalent to the condition that $\{S_\sigma(f) \mid \|f\|_{\Lambda_\alpha(X)} \leqq 1\}$ has uniformly absolutely continuous norm in $\Lambda(Y)$.

Remark. An alternate proof of Theorem 2 would be to use the "adjoint operator" $S_{\sigma'}$, where $\sigma' = [(Y_1', X_1'), (Y_2', X_2')]$. This method requires stronger hypotheses but would give that $(M^\circ(X), M^\circ(Y))$ is also weak compact intermediate for σ.

§5. **The compression operator and compact interpolation.** The upper and lower Boyd indices of a rearrangement invariant space X (see [1]) are given by

(5.1) $$\bar{\sigma}_X = \inf_{0<s<1} \theta(s) = \lim_{s\to 0} \theta(s)$$

and

(5.2) $$\underline{\sigma}_X = \sup_{s>1} \theta(s) = \lim_{s\to\infty} \theta(s)$$

where $\theta(s) = -\log h(s, X)/\log s$ and $h(s, X)$ is the operator norm on X of the compression operator

(5.3) $$\sigma_s(f)(t) = f(st)\chi_{(0,\ell)}(t).$$

These indices are known to satisfy

(5.4) $$0 \leqq \underline{\sigma}_X \leqq \underline{\gamma}_X \leqq \bar{\gamma}_X \leqq \bar{\sigma}_X \leqq 1$$

and

(5.5) $$\bar{\sigma}_{X'} = 1 - \underline{\sigma}_X , \qquad \underline{\sigma}_{X'} = 1 - \bar{\sigma}_X .$$

In [12] it was shown that Zippin's indices γ coincide with the Boyd indices if X contains L^∞ as a dense subset.

For the compression operator there obviously holds

(5.6) $$\sigma_{st} = \sigma_s\sigma_t$$

for all s and t if ℓ is infinite and for $t \geqq 1$ otherwise.

From the definitions (5.1) and (5.2) we have that for each positive ϵ there is a positive δ such that

$$h(1/s, X) \leqq s^{\bar{\sigma}_X+\epsilon} \quad \text{if} \quad 1/\delta < s, \tag{5.7}$$

$$h(1/s, X) \leqq s^{\underline{\sigma}_X-\epsilon} \quad \text{if} \quad 0 < s < \delta. \tag{5.8}$$

A rearrangement invariant space X is said to lie strictly between X_1 and X_2 if the Boyd indices of X lie between those of X_1 and X_2, *i.e.*, either $\bar{\sigma}_{X_1} < \underline{\sigma}_X$, $\bar{\sigma}_X < \underline{\sigma}_{X_2}$ or $\bar{\sigma}_{X_2} < \underline{\sigma}_X$, $\bar{\sigma}_X < \underline{\sigma}_{X_1}$. A pair (X, Y) is interior to an interpolation segment σ if (X, Y) is intermediate for σ and X (resp. Y) lies strictly between X_1 and X_2 (resp. Y_1 and Y_2).

In the next lemma which is essential for the proof of Theorem 3 we examine the conditions in which the following situation occurs:

(*) If T_n is a sequence of operators which are uniformly bounded as operators from X_i to Y_i, $i = 1, 2$, and which converge to zero in the operator norm in one of the two cases, then there is a subsequence such that $||T_{n_j}||_{B(X,Y)} \to 0$.

Remark. An equivalent formulation of (*) is:

(**) If $Z_1 \subset Z$, $Z_2 \subset W$ with continuous embeddings and V is a bounded subset of Z_1 which is sequentially precompact in Z_2, then V is sequentially precompact in Z

where we let $Z_1 = B(X_1, Y_1) \cap B(X_2, Y_2)$, $Z = B(X, Y)$, $Z_2 = B(X_1, Y_1) + B(X_2, Y_2)$, and $W = B(X_1 \cap X_2, Y_1 + Y_2)$. The space $X_1 \cap X_2$ is a Banach space when equipped with the norm $||f|| = \max (||f||_{X_1}, ||f||_{X_2})$. It is also easy to see that condition (**) holds when $W = Z_2$ and Z belongs to the interior of a scale connecting Z_1 with Z_2 (the logarithmically convex property of the norms of the scale (see [4])).

Lemma 7. *Suppose $\bar{\sigma}_{X_1} < \underline{\sigma}_{X_2}$ and $\bar{\sigma}_{Y_1} < \underline{\sigma}_{Y_2}$. If (X, Y) is interior to σ, then condition (*) holds.*

Proof. We let $||\cdot||_i$ be the norm in $B(X_i, Y_i)$ and $||\cdot||$ be the norm in $B(X, Y)$. Suppose $||T_n||_2 \leqq K$ and $||T_n||_1 \to 0$. We let

$$T_n' = c_n \sigma_{a_n} T_n$$

where

$$a_n = \left(\frac{||T_n||_2}{||T_n||_1}\right)^{1/(\alpha_2-\alpha_1)}, \qquad c_n = \left(\frac{||T_n||_2^{\alpha_1}}{||T_n||_1^{\alpha_2}}\right)^{1/(\alpha_2-\alpha_1)}. \tag{5.9}$$

In this case we let $\alpha_i = \underline{\sigma}_{Y_i} - \epsilon$. We may assume that $||T_n||_2$ is bounded away from zero, for if that is not the case, then there would be a subsequence T_{n_j} which would converge to zero in both norms, but (X, Y) is intermediate for σ so there is an A so that

$$||T||_{B(X,Y)} \leqq A \max ||T||_{B(X_i,Y_i)} \tag{5.10}$$

independent of T and hence $||T_{n_j}||_{B(X,Y)} \to 0$.

Since $||T_n||_2$ is bounded away from zero, we have that $a_n \to \infty$. By (5.8) we obtain

$$||T_n'||_i \leqq c_n h(a_n, Y_i)\ ||T_n||_i \leqq c_n a_n^{-\alpha_i}\ ||T_n||_i \leqq 1.$$

By (5.10) $||T_n'|| \leqq A$, for all n. Since $a_n \to \infty$, condition (5.6) implies

$$T_n = 1/c_n\ \sigma_{1/a_n} T_n'$$

and so

$$||T_n|| \leqq 1/c_n\ h(1/a_n, Y)A.$$

By (5.7) we have

$$||T_n|| \leqq A a_n^{\bar{\sigma}_Y+\epsilon}/c_n = A\ ||T_n||_2\ (1/a_n)^{\alpha_2-\bar{\sigma}_Y-\epsilon}$$
$$\leqq A \cdot K(1/a_n)^{\underline{\sigma}_{Y(2)}-\bar{\sigma}_Y-2\epsilon},$$

which goes to zero as n gets large since $\bar{\sigma}_Y < \underline{\sigma}_{Y_2}$.

On the other hand if $||T_n||_1 \leqq K$, $||T_n||_2 \to 0$, then we let $\alpha_i = \bar{\sigma}_{X_i} + \epsilon$ and $T_n' = c_n T_n \sigma_{a_n}$. In this case $a_n \to 0$, $T_n = (1/c_n) T_n' \sigma_{1/a_n}$, and we proceed as before.

Lemma 8. *Suppose* $\bar{\sigma}_{X_1} < \underline{\sigma}_{X_2}$, $\bar{\sigma}_{Y_2} < \underline{\sigma}_{Y_1}$, $\ell = \infty$, *and that* (X, Y) *is interior to* σ. *Then* (*) *holds.*

Proof. The argument follows as in Lemma 7 with $T_n' = c_n \sigma_{a_n} T_n$ and $\alpha_i = \bar{\sigma}_{X_i} + \epsilon$ (resp. $\bar{\sigma}_{Y_i} + \epsilon$), $i = 1, 2$, in the case $||T_n||_2 \to 0$, $||T_n||_1 \leqq K$ (resp. $||T_n||_1 \to 0$, $||T_n||_2 \leqq K$).

Lemma 9. *If* X *lies strictly between* X_1 *and* X_2, *then relation* (3.1) *holds.*

Proof. We assume without loss of generality that $\bar{\sigma}_{X_1} < \underline{\sigma}_X$, $\bar{\sigma}_X < \underline{\sigma}_{X_2}$. By (5.4) we have $\bar{\gamma}_{X_1} < \underline{\gamma}_X$, $\bar{\gamma}_X < \underline{\gamma}_{X_2}$. Suppose that $t \in (0, \ell)$. Then

$$\int_0^\ell F(s, t)\ ds/s \leqq (\max_{i=1,2} \varphi_Y(t)/\varphi_{Y_i}(t)) \int_0^\ell \min_{i=1,2} (\varphi_{X_i}(s)/\varphi_X(s))\ ds/s$$
$$= \text{const} \left(\int_0^\delta + \int_\delta^{\delta^{-1}} + \int_{\delta^{-1}}^\ell \right) \min_{i=1,2} (\varphi_{X_i}(s)/\varphi_X(s))\ ds/s.$$

The middle term is obviously bounded by the continuity of the fundamental functions. We choose δ as in (2.10) and (2.12), so

$$\int_0^\delta \min_{i=1,2} (\varphi_{X_i}(s)/\varphi_X(s))\ ds/s \leqq \int_0^\delta \min_{i=1,2} (s^{\underline{\gamma}_{X(i)}-\epsilon-\bar{\gamma}_X-\epsilon})\ ds/s$$
$$= \int_0^\delta s^{\underline{\gamma}_{X(2)}-\bar{\gamma}_X-2\epsilon}\ ds/s < \infty, \quad \text{where} \quad X(i) = X_i.$$

Proceed similarly for the remaining integral.

For a measurable subset E of $(0, \ell)$ define P_E (see [9]) by

$$P_E(f)(t) = \chi_E(t)f(t).$$

It is obvious that $||P_E||_{B(X,X)} \leqq 1$.

Theorem 3. *Suppose Y_1 and Y_2 have absolutely continuous norms and (X, Y) is interior to σ. If either*

(5.11) $\bar{\sigma}_{X_1} < \underline{\sigma}_{X_2}$, $\bar{\sigma}_{Y_1} < \underline{\sigma}_{Y_2}$ (i.e., *σ has positive slope*)

or

(5.12) $\ell = \infty$ *and* $\bar{\sigma}_{X_1} < \underline{\sigma}_{X_2}$, $\bar{\sigma}_{Y_2} < \underline{\sigma}_{Y_1}$ (i.e., *σ has negative slope*),

then (X, Y) is compact intermediate for σ.

Proof. Suppose for convenience that $T \in K(X_1, Y_1) \cap B(X_2, Y_2)$. Since X lies strictly between X_1 and X_2, Lemma 9 implies that condition (3.1) holds. Hence, by Lemma 1 we have that $V = \{Tf \mid ||f||_X \leqq 1\}$ is sequentially precompact in $\mathfrak{M}$. We need only show that V is uniformly absolutely continuous in Y, *i.e.*, show that

$$||P_{E_n}T||_{B(X,Y)} = \sup_{g \in V} ||\chi_{E_n} g||_Y \to 0$$

as $E_n \searrow \phi$. Since $T \in K(X_1, Y_1)$ and Y_1 has absolutely continuous norm, $V_1 = \{Tf \mid ||f||_{X_1} \leqq 1\}$ is uniformly absolutely continuous in Y_1, and so

$$||P_{E_n}T||_1 = \sup_{g \in V_1} ||\chi_{E_n} g|| \to 0.$$

Condition (5.11) (resp. (5.12)) gives that (*) holds true by Lemma 7 (resp. Lemma 8). Hence there is a subsequence such that $||P_{E_{n_j}}T|| \to 0$. But $||P_{E_n}T|| \searrow$ since $E_n \searrow \phi$, so $||P_{E_n}T|| \to 0$.

For convenience, we say that X is intermediate (compact intermediate, *etc.*) for (X_1, X_2) if (X, X) is intermediate for $[(X_1, X_1), (X_2, X_2)]$.

Corollary 1. *A necessary and sufficient condition for X to be compact intermediate for (L^p, L^q), $p < q$, is that*

(5.13) $$1/q < \underline{\sigma}_X, \ \bar{\sigma}_X < 1/p.$$

Proof. Sufficiency: By Theorem 1 of [1], condition (5.13) implies X is intermediate for (L^p, L^q). The case $p = 1$ or $q = \infty$ can be found in §5 of [7], so we assume $1 < p, q < \infty$. Since $\bar{\sigma}_Z = \underline{\sigma}_Z = 1/r$ when $Z = L^r$, Theorem 3 implies the assertion is true.

Necessity: Suppose X is compact intermediate for (L^p, L^q), but $\bar{\sigma}_X \geqq 1/p$. In Theorem 5 of [7] an operator T is constructed which is compact on L^∞, bounded on L^p, but is not compact on X (the construction for $\ell = \infty$ simplifies somewhat, but is essentially the same as for $\ell < \infty$). But then T would be compact on L^q since $p < q < \infty$, contradicting our assumption. Proceed similarly for the condition $\underline{\sigma}_X > 1/q$.

Corollary 2. *Suppose that L^∞ is a dense subset of each of the spaces X_1, X_2, and X. Assume further that*

$$\bar{\gamma}_{X_1} < \underline{\gamma}_X , \quad \bar{\gamma}_X < \underline{\gamma}_{X_2} ,$$

then X is compact intermediate for (X_1, X_2).

Proof. By the Boyd-Semenov-Zippin theorem, X is intermediate for (X_1, X_2). Theorem 1 of [12] implies that the Boyd and fundamental indices coincide for each of the three spaces. In order to apply Theorem 3 we note that separability of X_i is equivalent to the absolute continuity of its norm (see [8], Corollary 1, §3, Chapter I).

§6. **Interpolation with negative slope on a finite interval.** We assume throughout this section that X_i, Y_i', $i = 1, 2$, have absolutely continuous norms. We are then able to prove

Theorem 4. *Suppose that Y_i, $i = 1, 2$, have absolutely continuous norms, that $\ell < \infty$, and that $\bar{\sigma}_{X_1} < \underline{\sigma}_{X_2}$, $\bar{\sigma}_{Y_2} < \underline{\sigma}_{Y_1}$. If (X, Y) is interior to the segment σ, then (X, Y) is compact intermediate for σ.*

Since a necessary and sufficient condition for a Banach function space to be reflexive is that it and its associate space have absolutely continuous norms (see [8], Theorem 4, §2, Chapter I), Theorems 3 and 4 immediately give the following generalization of a theorem of Krasnosel''skii [3]:

Corollary 3. *Suppose that X_i, Y_i, $i = 1, 2$, are reflexive rearrangement invariant Banach function spaces and that (X, Y) is interior to σ, then (X, Y) is compact intermediate for σ.*

For the remainder of this section we assume that $\ell < \infty$ and that σ has negative slope, *i.e.*, $\bar{\sigma}_{X_1} < \underline{\sigma}_{X_2}$, $\bar{\sigma}_{Y_2} < \underline{\sigma}_{Y_1}$. The proof of one part of Theorem 4 is easy. Suppose that $T \in K(X_2, Y_2)$. We show that $T \in K(X, Y)$ when (X, Y) is interior to σ.

Lemma 10. *If $\bar{\gamma}_{Z_1} < \underline{\gamma}_{Z_2}$, then Z_1 is continuously embedded in Z_2.*

Proof. By (2.6) it suffices to show that $M(Z_1) \subset \Lambda(Z_2)$. Suppose $f \in M(Z_1)$. Then by the definition of the norm

$$f^*(t) \leqq \int_0^t f^*(s)\, ds/t \leqq \|f\|_{M(Z_1)}/\varphi_{Z_1}(t), \quad \text{all} \quad t.$$

So since $1/\varphi_{Z_1}$ decreases, we have

$$\|f\|_{\Lambda(Z_2)} \leqq \int_0^\ell d\varphi_{Z_2}(t)/\varphi_{Z_1}(t)\ \|f\|_{M(Z_1)} .$$

By relations (2.5), (2.10), and (2.12) we obtain

$$\int_0^{\ell} d\varphi_{Z_2}(t)/\varphi_{Z_1}(t) \leqq \int_0^{\ell} \varphi_{Z_2}(t)/\varphi_{Z_1}(t)\, dt/t$$

$$\leqq \text{const} \int_0^{\ell} t^{\underline{\gamma}_{Z(2)} - \epsilon - (\bar{\gamma}_{Z(1)} + \epsilon)}\, dt/t < \infty, \quad \text{where} \quad Z(i) = Z_i\,.$$

So since (X, Y) is interior to σ, property (5.4) implies that $\bar{\gamma}_X < \underline{\gamma}_{X_2}$ and $\bar{\gamma}_{Y_2} < \underline{\gamma}_Y$. By Lemma 10, $X \subset X_2$ and $Y_2 \subset Y$ with continuous embeddings, so $T \in K(X, Y)$.

The remaining part of the proof of Theorem 4, *i.e.*, $T \in K(X_1, Y_1) \cap B(X_2, Y_2)$ implies $T \in K(X, Y)$, is much harder, as indicated by Lemma 10, and requires several lemmas. For convenience, we denote by $\|\cdot\|$, $\|\cdot\|'$, $\|\cdot\|_i$, and $\|\cdot\|_i'$, $i = 1, 2$, the operator norms in the spaces $B(X, Y)$, $B(Y', X')$, $B(X_i, Y_i)$, and $B(Y_i', X_i')$, respectively.

Lemma 11. *Suppose $T \in B(X_1, Y_1) \cap B(X_2, Y_2)$. Then there is a unique operator T^a (in fact, the conjugate operator) which satisfies:*

(1) *For $i = 1, 2$,*

$$\int_0^{\ell} f\, T^a g\, dt = \int_0^{\ell} Tf\, g\, dt, \qquad f \in X_i\,, \quad g \in Y_i'\,.$$

(2) $T^a \in B(Y_1', X_1') \cap B(Y_2', X_2')$ *and* $\|T^a\|_i' = \|T\|_i$.

(3) $(\sigma_b)^a \in B(X', X')$ *and* $(\sigma_b)^a = 1/b\, \sigma_{1/b}$.

(4) $(P_F)^a \in B(X', X')$ *and* $(P_F)^a = P_F$.

Proof. Since X_1 has absolutely continuous norm, X_1^* is isometrically isomorphic to X_1'. But, for each $g \in Y_1'$, $G_1(\cdot) = \int_0^{\ell} gT(\cdot)dt$ belongs to X_1^*, so there is a unique function, $T_1^a g$ say, such that $\int fT_1^a g\, dt = \int gTf\, dt$ for all $f \in X_1$, $g \in Y_1'$. Similarly, we obtain T_2^a. We show that these operators coincide on $Y_1' \cap Y_2'$, so, we may in fact consider them as a single operator T^a. Suppose $g \in Y_1' \cap Y_2'$. Then since $\chi_E \in X_1 \cap X_2$, we have

$$\int_E T_1^a\, g\, dt = \langle \chi_E\,, T_1^a\, g\rangle = \langle T\chi_E\,, g\rangle$$

$$= \langle \chi_E\,, T_2^a\, g\rangle = \int_E T_2^a\, g\, dt.$$

Hence $T_1^a g = T_2^a g$ a.e.

To prove part (2) we use part (1) to obtain (where $Y(i)' = Y_i'$ and $X(i) = X_i$)

$$\|T^a\|_i' = \sup_{\|g\|_{Y(i)'} \le 1} \|T_g^a\|_{X'_i} = \sup_{\|g\|_{Y(i)'} \le 1}\ \sup_{\|f\|_{X(i)} \le 1} \langle T_g^a\,, f\rangle$$

$$= \sup_{\|f\|_{X(i)} \le 1}\ \sup_{\|g\|_{Y(i)'} \le 1} \langle g, Tf\rangle = \sup_{\|f\|_{X(i)} \le 1} \|Tf\|_{Y'_i}$$

$$= \|T\|_i\,, \qquad i = 1, 2.$$

Parts (3) and (4) are proved similarly, so we just prove part (3). Since $\sigma_b \varepsilon B(X, X)$, the integral $\int_0^\ell \sigma_b f g \, dt$ is finite for each $f \varepsilon X$, $g \varepsilon X'$. But

$$\int_0^\ell \sigma_b (f) \, g \, dt = (1/b) \int_0^\ell f \, \sigma_{1/b} \, g \, dt$$

by a change of variable $t \to t/b$. Hence the conjugate operator is equal $(1/b)\sigma_{1/b}$.

Lemma 12. *Suppose (X, Y) is intermediate for σ and $\varphi_X(0^+) = 0$. Then each operator S belonging to $B(Y_1', X_1') \cap B(Y_2', X_2')$ has an extension S_e satisfying:*

(1) $S_e \varepsilon B(Y', X')$;
(2) $(S\sigma_b)_e = S_e \sigma_b$;
(3) $(SP_F)_e = S_e P_F$;
(4) $\|S_e\|_{B(Y', X')} \leqq A \max_{i=1,2} \|S\|_{B(Y_i', X_i')}$, *all* S.

Proof. From Lemma 11 we have the existence of an operator S^a (Y_i' has absolutely continuous norm) which belongs to $B(X_i, Y_i)$, $i = 1, 2$. But (X, Y) is intermediate for σ, so S^a has a unique extension $(S^a)^-$ mapping X to Y. Moreover, there is a constant A independent of S, such that

$$\|(S^a)^-\| \leqq A \max_{i=1,2} \|S^a\|_i . \tag{6.1}$$

Let S_e be the conjugate operator of $(S^a)^-/X_\chi$, the restriction of $(S^a)^-$ to the closed subspace X_χ of functions in X with absolutely continuous norm. Since $(X_\chi)^* = X'$ (page 18 of [8]), we have $S_e \varepsilon B(Y^*, (X_\chi)^*) \subset B(Y', X')$. This along with (6.1) and Lemma 11 (2) gives

$$\|S_e\|' \leqq \|S_e\|_{B(Y^*, X')} = \|(S^a)^-\| \leqq A \max_{i=1,2} \|S^a\|_i$$
$$= A \max_{i=1,2} \|S\|_i'$$

thereby proving (4). Now we show that S_e is an extension of S. Suppose that $g \varepsilon Y_1' \cap Y'$. Since $\varphi_X(0^+) = 0$, $\chi_E \varepsilon X_\chi$ for each $E \subset (0, \ell)$, so

$$\langle S_e g, \chi_E \rangle = \langle g, (S^a)^- \chi_E \rangle = \langle g, S^a \chi_E \rangle$$
$$= \langle Sg, \chi_E \rangle$$

and hence

$$S_e g = Sg \text{ a.e.}$$

Proceed similarly if $g \varepsilon Y_2' \cap Y'$.

We show only (2) since the proof of (3) is the same. Suppose $g \varepsilon Y'$, $E \subset (0, \ell)$. Then

$$\langle (S\sigma_b)_e g, \chi_E \rangle = \langle g, (S\sigma_b)^a \chi_E \rangle$$
$$= \langle g, \sigma_b^a S^a \chi_E \rangle$$

$$= \langle \sigma_b g, S^a \chi_E \rangle$$
$$= \langle S_e \sigma_b g, \chi_E \rangle.$$

Hence $(S\sigma_b)_e g = S_e \sigma_b g$ a.e.

Lemma 13. Suppose $T \in K(X_1, Y_1) \cap B(X_2, Y_2)$, $E_n \searrow \phi$, and that (X, Y) is interior to σ. Then there is a subsequence $\{E_{n_j}\}$ such that

$$\|(T^a)_e P_{E_{n_j}}\|_{B(Y', X')} \to 0.$$

Proof. Since Y' is rearrangement invariant, we have

$$\|(T^a)_e P_{E_n}\|' = \|(T^a)_e P_{(0, mE_n)}\|'$$

so, it suffices to show that $\|(T^a)_e P_{(0,b_n)}\|' \to 0$ for some null sequence $\{b_n\}$. This follows since $mE_n \searrow 0$ and so there is a subsequence such that $mE_{n_j} \leqq b_n$ in which case

$$\|(T^a)_e P_{E_{n_j}}\|' \leqq \|(T^a)_e P_{(0,b_n)}\|'.$$

We let $S_n = T^a P_{(0, mE_n)}$. Then since $T \in K(X_1, Y_1)$, we have by Lemma 11, properties (4) and (2), that

$$\|S_n\|_1' = \|(P_{(0,mE_n)}T)^a\|_1' = \|P_{(0,mE_n)}T\|_1$$
$$= \sup_{\|f\|_{X(1)} \leq 1} \|\chi_{(0,mE_n)} Tf\|_{Y_1} \to 0, \quad \text{where} \quad X(1) = X_1 .$$

Similarly, $\|S_n\|_2' \leqq \|T\|_2$. We can assume that $\|S_n\|_2'$ is bounded away from zero for if that were not the case, then we could take a subsequence converging to zero and be through by applying (3) and (4) of Lemma 12. Now define

$$a_n = \left(\frac{\|S_n\|_1'}{\|S_n\|_2'}\right)^{1/(\alpha_1 - \alpha_2)}, \qquad c_n = \left(\frac{(\|S_n\|_1')^{\alpha_2}}{(\|S_n\|_2')^{\alpha_1}}\right)^{1/(\alpha_1 - \alpha_2)}$$

where $\alpha_i = \underline{\sigma}_{Y_i'} - \epsilon$, ϵ chosen so that $\epsilon < (\underline{\sigma}_Y - \bar{\sigma}_{Y_2})/4$. But by (5.5), $\bar{\sigma}_{Y_2} < \underline{\sigma}_{Y_1}$ implies $\bar{\sigma}_{Y_1'} < \underline{\sigma}_{Y_2'}$ and so $a_n \to \infty$. Let

$$S_n' = c_n S_n \sigma_{a_n} .$$

Then by (5.8)

$$\|S_n'\|_i' \leqq c_n \|S_n\|_i' h(a_n, Y_i')$$
$$\leqq c_n \|S_n\|_i' a_n^{-\alpha_i} = 1.$$

By (4) of Lemma 12 we have

$$\|(S_n')_e\|' \leqq A, \tag{6.2}$$

but by (2) of the same lemma

$$(S_n)_e P_{(0, t/a_n)} = (S_n)_e \sigma_{a_n} \sigma_{1/a_n} = c_n^{-1} (S_n')_e \sigma_{1/a_n} . \tag{6.3}$$

If we let $b_n = \min (mE_n, \ell/a_n)$, then by Lemma 12 (3), relation (6.3), and (6.2) we obtain

$$\begin{aligned} \|(T^a)_e P_{(0,b_n)}\|' &\leqq A c_n^{-1} a_n^{\bar{\sigma}_{Y'}+\epsilon} \\ &= A \, \|S_n\|_2' \, a_n^{\bar{\sigma}_{Y'}+\epsilon-\alpha^2} \\ &\leqq A \, \|T\|_2 \, a_n^{\bar{\sigma}_{Y'}-\underline{\sigma}_{Y(2)'}+2\epsilon} \to 0, \quad \text{where} \quad Y(2)' = Y_2'. \end{aligned}$$

Lemma 14. Suppose (X, Y) is interior to σ. Then

$$\|P_E T\|_{B(X,Y)} \leqq \|(T^a)_e P_E\|_{B(Y',X')}.$$

Proof. First we claim that $S = ((T^a)_e P_E/(Y')_X)^a$ coincides with $P_E T$ on X. We prove this by showing that if $f \in X \cap X_i$ ($i = 1$ or 2), then $Sf = P_E Tf$. Since (X, Y) is intermediate for σ, the extension to X must be unique. So, let $F \subset (0, \ell)$ and $f \in X \cap X_1$ for convenience. Then $\chi_F \in (Y')_X \cap Y_1'$, so Lemmas 11 and 12 give

$$\begin{aligned} \langle Sf, \chi_F \rangle &= \langle f, (T^a)_e P_E \chi_F \rangle \\ &= \langle f, T^a P_E \chi_F \rangle \\ &= \langle P_E T f, \chi_F \rangle. \end{aligned}$$

Hence, we have

$$\begin{aligned} \|P_E T\| &= \sup_{\|f\|_X \leq 1} \|Sf\|_Y = \sup_{\|f\|_X \leq 1} \sup_{\substack{g \in (Y')_X \\ \|g\|_{Y'} \leq 1}} \langle Sf, g \rangle \\ &= \sup_{\substack{g \in (Y')_X \\ \|g\|_{Y'} \leq 1}} \sup_{\|f\|_X \leq 1} \langle f, (T^a)_e P_F g \rangle \leqq \sup_{\|g\|_{Y'} \leq 1} \|(T^a)_e P_E g\|_{X'} \\ &\leqq \|(T^a)_e P_E\|'. \end{aligned}$$

We can now complete the proof of Theorem 4. As in §5, Lemma 1 implies that it suffices to show that $V = \{Tf \mid \|f\|_X \leqq 1\}$ has uniformly absolutely continuous norm in Y. This is equivalent to showing that

$$\|P_{E_n} T\|_{B(X,Y)} \to 0$$

for each sequence $E_n \searrow \phi$. By Lemmas 13 and 14, we see that for some sequence

$$\|P_{E_{n(i)}} T\| \to 0.$$

But, since $E_n \searrow \phi$, $\|P_{E_n} T\| \searrow$ and the result follows.

Acknowledgment. The author would like to express his sincere gratitude to Professor George Lorentz for his many thoughtful suggestions.

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This research was partially sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant No. AFOSR 71-2130; by National Science Foundation Grant GP-23566 and by a National Science Foundation Center of Excellence Grant to L. S. U.

Part of this research appeared in the author's doctoral dissertation written at the University of Texas under the supervision of Professor G. G. Lorentz.

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Date communicated: August 4, 1972