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Spaces $\Lambda_\alpha(X)$ and Interpolation*

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For two pairs of rearrangement invariant spaces $\sigma = [(X_1, Y_1), (X_2, Y_2)]$ we give necessary and sufficient conditions for pairs (X, Y) to be weak intermediate for σ, i.e., each operator which is of weak types (X_i, Y_i), $i = 1, 2$, also maps X boundedly to Y. Spaces $\Lambda_\alpha(X)$ are introduced and are shown to have many of the properties that characterize Lorentz L^{pq} spaces. Necessary and sufficient conditions in terms of a simple function $F(s, t)$ are given in order that $(\Lambda_\alpha(X), \Lambda_\alpha(Y))$ be weak intermediate for σ. Other properties of the function $F(s, t)$ yield sufficient conditions and necessary conditions for interpolation theorems.

1. INTRODUCTION

The purpose of this paper is to provide interpolation theorems for a general interpolation segment $\sigma = [(X_1, Y_1), (X_2, Y_2)]$, where all four spaces are arbitrary, rearrangement-invariant Banach function spaces. Two developments are necessary to proceed: (a) a general form of Calderón's weak interpolation theorem, and (b) spaces that play in the general context the same role as the Lorentz L^{pq} spaces. These ideas are developed in Sections 4 and 3, respectively; Section 5 combines these ideas in order to obtain some concrete interpolation theorems.

A Banach space X of real-valued Lebesgue measurable functions on a possibly infinite interval $I = (0, l)$ is said to be a Banach function space over I if

$$|g| \leqslant |f| \quad \text{a.e. and } f \in X, \text{ then } g \in X \text{ and } \|g\| \leqslant \|f\|. \tag{1.1}$$

$$f_n \in X,\ \|f_n\| \leqslant M, \text{ and } 0 \leqslant f_n \uparrow f \text{ a.e., then } f \in X \text{ and } \|f\| \leqslant M. \tag{1.2}$$

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580/11/4-7

A Banach function space X is said to be rearrangement invariant if $m\{s \mid |g(s)| > \epsilon\} < +\infty$ for each $\epsilon > 0$ and $g \in X$, and the following condition is satisfied:

if $g \in X$ and g' is any function on I equimeasurable with g, then $g' \in X$ and $\| g' \| = \| g \|$.

For spaces of this type we will write, in short, X is a rearrangement-invariant space.

Examples of rearrangement-invariant spaces include the Lebesgue L^p spaces, the Lorentz Λ, M, and L^{pq} spaces [6], and Orlicz spaces.

Let X_1, Y_1, X_2, Y_2 be two pairs of rearrangement-invariant spaces. If each linear operator which is a bounded operator from X_i to Y_i, $i = 1, 2$, has a unique extension to a bounded operator from X to Y, then the pair (X, Y) is said to be strong intermediate for the interpolation segment $\sigma = [(X_1, Y_1), (X_2, Y_2)]$. The pair (X, Y) is said to be weak intermediate for the interpolation segment σ if each linear operator which is a weak-type (X_i, Y_i) operator, $i = 1, 2$, has a unique extension to a bounded operator from X to Y.

We do not aim at the maximal generality. One could replace in our theorems the interval I by any nonatomic σ-finite measure space, and linear operators by quasilinear ones. See Calderón [3].

In Section 2 we give properties of the fundamental function ϕ_X of a rearrangement-invariant space X and relate weak-type operators to bounded operators of Lorentz Λ and M spaces (see Lorentz [6, Section 3.5]). Section 3 deals with the spaces $\Lambda_\alpha(X)$ which are generalizations of the Lorentz L^{pq} spaces (see [2, 3, 7], or [11]). These may be regarded as a combination of an L^p space with Lebesgue measure and of an L^q space on $(0, l)$ with measure dt/t. One obtains $\Lambda_\alpha(X)$ if one puts $\alpha = 1/q$ and replaces L^p by an arbitrary rearrangement-invariant space X. We show in Section 3 that $\Lambda_\alpha(X)$ is reflexive for all $0 < \alpha < 1$; in fact, the conjugate space, modulo renorming, is represented as another $\Lambda_\beta(Y)$ space. In Section 4 we define Calderón's operator and prove its maximality as a weak-type (X_i, Y_i) operator (Theorem 4.5). Theorem 4.7 gives a necessary and sufficient condition for a pair (X, Y) to be weak intermediate for an interpolation segment. In [16] Zippin states Theorem 4.7 under the following conditions: L^∞ is dense in all spaces X_1, X_2, Y_1, Y_2, X, Y; the fundamental indices satisfy $\bar{\gamma}_{X_1} < \gamma_{X_2}$; and the fundamental functions of the spaces X_1, X_2, Y_1, Y_2 have the property that for some $\delta > 0$, $N > 0$

$$\phi_{X_1}(s)/\phi_{Y_1}(t) \leqslant N \cdot \phi_{X_2}(s)/\phi_{Y_2}(t) \qquad \text{if} \quad 0 < s < \delta t$$

and

$$\phi_{X_2}(s)/\phi_{Y_2}(t) \leqslant N \cdot \phi_{X_2}(s)/\phi_{Y_1}(t) \qquad \text{if} \quad s \geqslant \delta t.$$

We do not require these conditions, but the idea of the proof is still essentially Calderón's [3].

We combine the results of Sections 3 and 4 in Section 5. Here the question of interpolation is in many cases reduced to calculable criteria in terms of the behavior of the function $F(s, t)$ which appears for the first time in Lorentz–Shimogaki [8]. The function F is determined by the six fundamental functions of the spaces involved; no deeper properties of the spaces appear in these theorems. Theorems 5.1 and 5.2 give necessary and sufficient conditions for the pairs $(\Lambda(X), \Lambda(Y))$ and $(M(X), M(Y))$, respectively, to be weak intermediate for the interpolation segment σ. A sufficient condition for (X, Y) to be weak intermediate (and hence strong intermediate) for σ is given in Theorem 5.9, while a necessary condition for (X, Y) to be weak intermediate for σ appears in Theorem 5.7. Theorem 5.6 is an interpolation theorem involving the spaces $\Lambda_\alpha(X)$ which generalizes Calderón's result that (L^{pr}, L^{qr}), $1 \leqslant r \leqslant \infty$, is weak intermediate for $\sigma = [(L^{p_1}, L^{q_1}), (L^{p_2}, L^{q_2})]$.

In the last section, 6, we compare the methods of Calderón with those of Lorentz–Shimogaki (quasiorder $\prec$) and Peetre [functionals $K(t, f)$]. If the interpolation segment is

$$\bar{\sigma} = [(\Lambda(X_1), M(Y_1)), (\Lambda(X_2), M(Y_2))]$$

these methods yield the same intermediate spaces. Also we outline the proof that the spaces $\Lambda_\alpha(X)$ form a scale, in the sense of Krein and Petunin [5], which connects $\Lambda(X)$ and $M^0(X)$, the norm closure of L^∞ in $M(X)$.

We will denote by X' the Banach function space consisting of all measurable functions g on I such that

$$\| g \|_{X'} = \sup_{\|f\|_X \leqslant 1} \left| \int_0^l f(s)\, g(s)\, ds \right| \tag{1.3}$$

is finite, and we denote by X^* the conjugate space of X. We will need the following representation of the norm of X given by Lorentz and Luxemburg [10]:

$$\| f \|_X = \sup_{\|g\|_{X'} \leqslant 1} \left| \int_0^l g(s)\, f(s)\, ds \right|. \tag{1.4}$$

This gives immediately that $X'' = X$.

Suppose X is a rearrangement-invariant space and let $f \in X$. Then, by definition, $m\{x \,\|f(x)| > \epsilon\}$ is finite for $\epsilon > 0$. We denote by f^*, the unique, right-continuous decreasing function on I which is equimeasurable with $|f|$. We then have that $\|f^*\|_X = \|f\|_X$; in fact,

$$\|f\|_X = \sup_{\|g\|_{X'} \leqslant 1} \int_0^l f^*(s)\, g^*(s)\, ds. \tag{1.5}$$

We can now see that X' is a rearrangement-invariant space if X is.

For a function f, if f^* exists, we define $f^{**}(t) = \int_0^t f^*(s)\, ds/t$. We then write $h \prec f$ if $h^{**}(t) \leqslant f^{**}(t)$, $0 < t < l$. By (1.5), if $h \prec f$ and $f \in X$, then $h \in X$ and $\|h\|_X \leqslant \|f\|_X$.

We denote by $X_1 + X_2$ the sum of two rearrangement-invariant spaces equipped with the norm

$$\|f\|_{X_1+X_2} = \inf_{\substack{f=f_1+f_2 \\ f_i \in X_i}} \{\|f_1\|_{X_1} + \|f_2\|_{X_2}\}. \tag{1.6}$$

It is clear that $X_1 + X_2$ becomes a rearrangement-invariant space with this norm.

The fundamental function ϕ_X of a rearrangement-invariant space X is defined by

$$\phi_X(t) = \|\chi_{(0,t)}\|_X \qquad 0 < t < l,$$

where $\chi_{(0,t)}$ is the characteristic function of the interval $(0, t)$.

The following proposition appears in the literature:

PROPOSITION. 1.1 *Let X be a rearrangement-invariant space, then*

$$\phi_X(t) \cdot \phi_{X'}(t) = t \qquad 0 < t < l; \tag{1.7}$$

$$\phi_X(t) \text{ is a continuous increasing function on } (0, l); \tag{1.8}$$

For each $\epsilon > 0$, ϕ_X is absolutely continnous on $[\epsilon, l)$.
Moreover, there holds $d\phi_X(t)/dt \leqslant \phi_X(t)/t$, a.e. (1.9)

X can be equivalently renormed, by $\| \,\|_0$, say,
so that the resulting fundamental function is concave and satisfies (1.10)

$$\phi_X(t) \leqslant \phi_{X_0}(t) \leqslant 2\phi_X(t) \qquad 0 < t < l.$$

Using averages, Semenov proves (1.7) in [13]. Statement (1.8) is an immediate consequence of (1.7) and the definition of ϕ_X. Statement (1.9) is given in [16]; indeed, differentiating $\phi_X(t) = t/\phi_{X'}(t)$, we can easily get $d\phi_X(t)/dt \leqslant \phi_X(t)/t$ a.e., since $\phi_{X'}(t)$ increases. Statement (1.10) is given in [12] and a proof appears in [16]. An elementary proof is given in [15].

We note that if X is any of the spaces L^p, $\Lambda_{1/p}$, $M_{1-1/p}$, L^{pq}, or $L^p(\log^+ L)^q$, then $\phi_X(t) = t^{1/p}$.

Of fundamental importance for us are the two classes of rearrangement-invariant spaces given by

$X \in \mathscr{U}$ if for some $\alpha < 1$ there is a pair of positive constants C and δ such that (1.11)

$$\phi_X(u)/\phi_X(v) \leqslant C(u/v)^\alpha \qquad \text{if} \quad v/u \leqslant \delta.$$

$X \in \mathscr{L}$ if for some $\beta > 0$, there is a pair of positive constants C and δ such that (1.12)

$$\phi_X(v)/\phi_X(u) \leqslant C(v/u)^\beta \qquad \text{if} \quad v/u \leqslant \delta.$$

PROPOSITION 1.2. *$X' \in \mathscr{L}$ if and only if $X \in \mathscr{U}$.*

Proof. Suppose $X' \in \mathscr{L}$, then (1.12) holds. Let $\alpha = 1 - \beta$, then

$$u/v\phi_{X'}(v)/\phi_{X'}(u) \leqslant C(v/u)^{\beta-1} \qquad \text{if} \quad v/u \leqslant \delta.$$

Remark 1.3. If $\bar{X}$ is the space X with an equivalent norm, then $\bar{X} \in \mathscr{U}$ if and only if $X \in \mathscr{U}$.

Remark 1.4. The fundamental indices

$$\gamma_X = \sup_{0<t<1} \log M(t, X)/\log t = \lim_{t\to 0} \log M(t, X)/\log t$$

and

$$\bar{\gamma}_X = \inf_{t>1} \log M(t, X)/\log t = \lim_{t\to\infty} \log M(t, X)/\log t,$$

where $M(t, X) = \sup_s \phi_X(st)/\phi_X(s)$, were introduced by Zippin [16]. We note that $X \in \mathscr{U}$ is equivalent to $\bar{\gamma}_X < 1$, while $0 < \gamma_X$ is equivalent to $X \in \mathscr{L}$. The following is known about the fundamental indices of a rearrangement invariant space:

$$0 \leqslant \gamma_X \leqslant \bar{\gamma}_X \leqslant 1, \tag{1.13}$$

$$\bar{\gamma}_X = 1 - \gamma_{X'}. \tag{1.14}$$

It is in the sense that $\bar{\gamma}_X < 1$, that we say X is not "close" to L^1.

2. ϕ_X and Weak (X, Y) Operators

We assume throughout this section as well as the rest of the paper that X has a concave fundamental function ϕ_X. This is no restriction by (1.10); the proofs would have to be modified slightly and constants would have to be introduced in certain inequalities. We first define extremal rearrangement-invariant spaces $\Lambda(X)$ and $M(X)$ (see [12]), and we show that weak (X, Y) operators are identical with bounded operators from $\Lambda(X)$ to $M(Y)$ except in extreme cases of Y, i.e., when Y does not belong to the class $\mathscr{U}$.

We define $\Lambda(X)$ to be the set of all I-measurable functions f such that f^* exists and $\|f\|_{\Lambda(X)} = \int_0^l f^*(s)\, d\phi_X(s)$ is finite. The rearrangement-invariant space $M(X)$ is the set of all f such that f^* exists and $\|f\|_{M(X)} = \sup_{t>0} f^{**}(t)\phi_X(t)$ is finite. $\Lambda(X)$ and $M(X)$ are Lorentz Λ and M spaces, respectively, even in the case $\phi_X(0^+) \neq 0$. In [12] and [16] it is shown that

$$\Lambda(X) \subset X \subset M(X) \tag{2.1}$$

with continuous embeddings. Indeed, considering the covariant decomposition of positive decreasing simple functions and the covariantly additive property of the norm of $\Lambda(X)$, we get $\Lambda(X)$ is continuously contained in X by passing to the limit f^* and using (1.2). We give more detail in the proof of the second containment of (2.1) since it does not appear in the literature. We make use of Hölder's inequality for function spaces [10]. Suppose $f \in X$, then

$$\begin{aligned}\|f\|_{M(X)} &= \sup_t \int_0^t f^*(s)\, ds/\phi_{X'}(t) = \sup_t \left\{\int_0^l \chi_{(0,t)}(s) f^*(s)\, ds/\phi_{X'}(t)\right\}\\ &\leqslant \sup_t \{\|f\|_X \cdot \|\chi_{(0,t)}\|_{X'}/\phi_{X'}(t)\} = \|f\|_X .\end{aligned}$$

A linear operator T is said to be of weak type (X, Y) if T maps $\Lambda(X)$ into the set of measurable functions on I and for some constant C satisfies

$$\sup_t \{(Tf)^*(t)\, \phi_Y(t)\} \leqslant C\, \|f\|_{\Lambda(X)} \tag{2.2}$$

for each $f \in \Lambda(X)$. The smallest constant C such that (2.2) holds is denoted $\|T\|_{w(X,Y)}$. If we denote by $\mathscr{M}$ the topological vector space of measurable functions on I with the topology of convergence in measure on subsets of finite measure, and if T is an operator of weak type (X, Y), then T is a continuous operator from $\Lambda(X)$ to $\mathscr{M}$.

LEMMA 2.1. *If $Y \in \mathscr{U}$, then for some constant $C(Y)$*

$$\phi_{Y'}(t) \leqslant \int_0^t 1/\phi_Y(s)\,ds \leqslant C(Y)\phi_{Y'}(t) \tag{2.3}$$

for $t \in I$.

Proof. Since $\phi_Y(s)$ increases,

$$\phi_{Y'}(t) = \int_0^t ds/\phi_Y(t) \leqslant \int_0^t 1/\phi_Y(s)\,ds.$$

In order to prove the right-hand side of (2.3), we use (1.11) since $Y \in \mathscr{U}$. We can assume $\delta < 1$ and let $r = t/\delta$. Then,

$$\begin{aligned}\int_0^t 1/\phi_Y(s)\,ds &= \phi_{Y'}(t) \cdot \phi_Y(t)/(t\phi_Y(r)) \cdot \int_0^{\delta r} \phi_Y(r)/\phi_Y(s)\,ds \\ &\leqslant C \cdot \phi_{Y'}(t)/t \int_0^{\delta r} (r/s)^\alpha\,ds \leqslant C/(1-\alpha) \cdot r^\alpha t^{1-\alpha}\phi_{Y'}(t)/t \\ &= C(Y)\phi_{Y'}(t)\end{aligned}$$

since $\phi_Y(t)/\phi_Y(r) \leqslant 1$ and $s/r \leqslant \delta$.

THEOREM 2.2. *We let* $\|f\|^*_{M(Y)} = \sup_t\{f^*(t)\phi_Y(t)\}$. *If $Y \in \mathscr{U}$, then*

$$\|f\|^*_{M(Y)} \leqslant \|f\|_{M(Y)} \leqslant C(Y)\|f\|^*_{M(Y)} \tag{2.4}$$

for each $f \in M(Y)$.

Proof. Since f^* decreases, the left-hand side of (2.4) follows from

$$f^*(t) = f^*(t)\int_0^t ds/t \leqslant \int_0^t f^*(s)\,ds/t = f^{**}(t).$$

On the other hand, suppose $\|f\|^*_{M(Y)}$ is finite, then

$$f^*(s) \leqslant \|f\|^*_{M(Y)}/\phi_Y(s). \tag{2.5}$$

On integrating (2.5) and applying Lemma 2.1, we obtain

$$\begin{aligned}f^{**}(t) &\leqslant \|f\|^*_{M(Y)}\int_0^t 1/\phi_Y(s)\,ds/t \\ &\leqslant C(Y)\|f\|^*_{M(Y)}/\phi_Y(t).\end{aligned} \tag{2.6}$$

Multiplying both sides of (2.6) by $\phi_Y(t)$ and taking the supremum over all t we obtain

$$\|f\|_{M(Y)} \leqslant C(Y)\|f\|^*_{M(Y)}.$$

COROLLARY 2.3. *If $Y \in \mathscr{U}$, then weak-type (X, Y) operators are equivalent to bounded operators from $\Lambda(X)$ to $M(Y)$.*

THEOREM 2.4. *If $X \in \mathscr{L}$, then X can be equivalently renormed such that the resulting fundamental function ϕ_X has the following properties:*

$$\phi_X \text{ is concave,} \tag{2.7}$$

$$t\, d\phi_X(t)/dt \text{ is increasing and concave,} \tag{2.8}$$

$$\text{there is a constant } C \text{ such that} \tag{2.9}$$

$$C^{-1}\phi_X(t)/t \leqslant d\phi_X(t)/dt \leqslant \phi_X(t)/t \qquad t \in I.$$

Proof. Since $X \in \mathscr{L}$, then $X' \in \mathscr{U}$. Applying Lemma 2.1 with $Y = X'$ and setting $\bar{\phi}(t) = \int_0^t \phi_X(s)/s\, ds$, we have

$$\phi_X(t) \leqslant \bar{\phi}(t) \leqslant C\phi_X(t) \qquad t \in I. \tag{2.10}$$

Now we set

$$\|f\|_{\bar{X}} = \max\{\|f\|_X, \sup_t\{f^{**}(t)\bar{\phi}(t)\}\}.$$

Using the continuous embedding (2.1) and (2.10), it is easy to see that

$$\|f\|_X \leqslant \|f\|_{\bar{X}} \leqslant C\|f\|_X.$$

Also, it is clear that $\phi_{\bar{X}}(t) = \bar{\phi}(t)$, $t \in I$.

(2.7) follows since $d\bar{\phi}(t)/dt = \phi_X(t)/t$ decreases. Since $td\bar{\phi}(t)/dt = \phi_X(t)$, statement (2.8) follows. (2.9) is a combination of (2.7) and the equivalent renorming.

THEOREM 2.5. *Assume $Y \in \mathscr{U}$. A necessary and sufficient condition for an operator to be of weak type (X, Y) is that T is a continuous operator from $\Lambda(X)$ into the space $\mathscr{M}$ of measurable functions and for some constant C satisfies*

$$\sup_t (T\chi_E)^*(t)\,\phi_Y(t) \leqslant C\phi_X(mE), \tag{2.11}$$

for each measurable $E \subset I$.

Proof. The necessity of (2.11) is obvious. To prove the sufficiency, suppose that $f_n \uparrow f^*$, where $\{f_n\}$ is a sequence of positive decreasing simple functions. But then f_n converges to f^* in $\Lambda(X)$, so Tf_n converges to Tf^* in measure on subsets of I of finite measure. This implies $(Tf_n)^*$ converges pointwise to $(Tf^*)^*$. By the properties (1.1) and (1.2) of a Banach function space, it suffices to prove the theorem for f a positive decreasing simple function, say $f(t) = \sum_{i=1}^n a_i\chi_{(0,s_i)}(t)$, where each $a_i \geqslant 0$. Applying Theorem 2.2, statement (2.11), and the covariantly additive property of the norm of $\Lambda(X)$, we obtain

$$\begin{aligned}\sup_t (Tf)^*(t)\,\phi_Y(t) \leqslant \| Tf\|_{M(Y)} &\leqslant \sum_{i=1}^n a_i \| T\chi_{(0,s_i)}\|_{M(Y)} \\ &\leqslant C(Y)\sum_{i=1}^n a_i\{\sup_t (T\chi_{(0,s_i)})^*(t)\,\phi_Y(t)\} \\ &\leqslant C\cdot C(Y)\cdot\sum_{i=1}^n a_i\phi_X(s_i) \\ &= C\cdot C(Y)\cdot\int_0^l f(s)\,d\phi_X(s) = C\cdot C(Y)\cdot\|f\|_{\Lambda(X)}.\end{aligned}$$

3. Spaces $\Lambda_\alpha(X)$

We introduce the spaces $\Lambda_\alpha(X)$ and show that they possess many of the properties that characterize the Lorentz L^{pq} spaces. We begin with Hardy's inequalities (Lemmas 3.1 and 3.6), proceed to embedding and norm properties (Corollary 3.3 and Proposition 3.4), and then conclude with duality results (Theorem 3.10 and Remark 3.11).

We assume throughout this section that $X \in \mathscr{L} \cap \mathscr{U}$. The space $\Lambda_\alpha(X)$, $0 < \alpha \leqslant 1$, is the set of all I-measurable functions f such that f^* exists and

$$\|f\|_{\Lambda_\alpha(X)} = \left\{\int_0^l [f^{**}(t)\,\phi_X(t)]^{1/\alpha}\,\frac{dt}{t}\right\}^\alpha \tag{3.1}$$

is finite.

The spaces $\Lambda_\alpha(X)$ appear in [7] where spaces called $\Lambda_{\psi,1/\alpha}$ are considered with norm

$$\left\{\int_0^\infty [f^*(s)]^{1/\alpha}\,\psi(s)\,ds\right\}^\alpha.$$

If one replaces f^{**} in $\Lambda_\alpha(X)$ by f^*, which leads to an equivalent norm as we show below, and assumes, that $\phi_X(t)^{1/\alpha}/t$ decreases, then one is dealing with a $\Lambda_{\psi,1/\alpha}$ space.

It is clear that $\Lambda_\alpha(X)$ is a rearrangement-invariant space for each X and $0 < \alpha \leqslant 1$. Suppose that X is equivalently renormed and is denoted by $\bar{X}$. By (3.1) it is easy to see that $\Lambda_\alpha(\bar{X}) = \Lambda_\alpha(X)$ with equivalent norms.

From the following lemma emerge the embedding results mentioned above.

LEMMA 3.1. *The functional*

$$\|f\|^*_{\Lambda_\alpha(X)} = \left\{\int_0^l [f^*(s)\,\phi_X(s)]^{1/\alpha} \frac{ds}{s}\right\}^\alpha$$

is equivalent to the norm of $\Lambda_\alpha(X)$: for some constant C

$$\|f\|^*_{\Lambda_\alpha(X)} \leqslant \|f\|_{\Lambda_\alpha(X)} \leqslant C\,\|f\|^*_{\Lambda_\alpha(X)}\,. \tag{3.2}$$

Proof. The left inequality holds since $f^*(t) \leqslant f^{**}(t)$. For the right inequality we can first assume that there exist numbers $0 < a \leqslant b < l$ such that f^* is constant on $(0, a)$ and zero on (b, l) by the monotone convergence theorem. By Proposition 1.2, $X' \in \mathscr{L}$ since $X \in \mathscr{U}$. Applying Theorem 2.4, we denote by Y the space X' equivalently renormed and satisfying (2.7)–(2.9). If we let $F(t) = \int_0^t f^*(s)\,ds$, then

$$\begin{aligned}\int_0^l [f^{**}(t)\,\phi_X(t)]^{1/\alpha}\frac{dt}{t} &= \int_0^l \left[\frac{F(t)}{\phi_{X'}(t)}\right]^{1/\alpha}\frac{dt}{t}\\ &\leqslant \text{const} \int_0^l \left[\frac{F(t)}{\phi_Y(t)}\right]^{1/\alpha}\frac{dt}{t}\\ &\leqslant \text{const} \int_0^l \left[\frac{F(t)}{\phi_Y(t)}\right]^{1/\alpha}\frac{d\phi_Y(t)}{\phi_Y(t)}\\ &= -\,\text{const} \int_0^l F(t)^{1/\alpha}\,d[\phi_Y(t)^{-1/\alpha}],\end{aligned} \tag{3.3}$$

since Y satisfies (2.9). By (1.12) we have $\phi_X(0^+) = 0$ since $X \in \mathscr{L}$. We apply integration by parts to the last integral in (3.3). Since

$$\begin{aligned}-[F(t)/\phi_Y(t)]^{1/\alpha}\,\Big|_{0^+}^{+\infty} &\leqslant [F(t)/\phi_Y(t)]^{1/\alpha}\,\Big|_{t=0^+} \leqslant \text{const}[F(t)/\phi_{X'}(t)]^{1/\alpha}\,\Big|_{t=0^+}\\ &= \text{const}[f^{**}(t)\,\phi_X(t)]^{1/\alpha}\,\Big|_{t=0^+}\\ &= \text{const}[f^*(a)\phi_X(0^+)]^{1/\alpha}\end{aligned}$$

we obtain

$$\int_0^l [f^{**}(t)\,\phi_X(t)]^{1/\alpha}\frac{dt}{t}$$

$$\leqslant \text{const}\int_0^l \phi_Y(t)^{-1/\alpha}F(t)^{1/\alpha-1}f^*(t)\,dt$$

$$\leqslant \text{const}\int_0^l f^{**}(t)^{1/\alpha-1}f^*(t)\,\phi_X(t)^{1/\alpha}\frac{dt}{t}$$

$$\leqslant \text{const}\left\{\int_0^l [f^{**}(t)\,\phi_X(t)]^{1/\alpha}\frac{dt}{t}\right\}^{1-\alpha}\left\{\int_0^l [f^*(t)\,\phi_X(t)]^{1/\alpha}\frac{dt}{t}\right\}^{\alpha} \quad (3.4)$$

by the inequality $t\cdot\phi_Y(t)^{-1} \leqslant \text{const}\,\phi_X(t)$ and Hölder's inequality. If we can show that the first factor is finite, then we will be done since we can divide (3.4) by that factor and get the desired inequality. Since $f^*(a)\phi_X(b) \geqslant f^{**}(t)\,\phi_X(t)$ when $t\in(a,b)$, then $\int_a^b [f^{**}(t)\,\phi_X(t)]^{1/\alpha}(dt/t)$ is finite. We only need to check the two tails of the integral for finiteness. Since $f^{**}(t) = F(b)/t$ for $t\in(b, l)$ we obtain

$$\int_b^l [f^{**}(t)\,\phi_X(t)]^{1/\alpha}\frac{dt}{t} = F(b)^{1/\alpha}\int_b^l \phi_{X'}(t)^{-1/\alpha}\frac{dt}{t}$$

$$\leqslant \text{const}\int_b^l \phi_Y(t)^{-1/\alpha}\frac{d\phi_Y(t)}{\phi_Y(t)}$$

$$\leqslant \text{const}\,\phi_Y(t)^{-1/\alpha}\,\Big|_b^l < +\infty.$$

Since $X\in\mathscr{L}$, we denote by $\bar{X}$ the space X which is renormed according to Theorem 2.4. But $f^{**}(t) = f^*(a)$ for $t\in(0, a)$, so we have

$$\int_0^a [f^{**}(t)\,\phi_X(t)]^{1/\alpha}\frac{dt}{t} \leqslant \text{const}\,f^*(a)^{1/\alpha}\int_0^a \phi_X(t)^{1/\alpha}\frac{dt}{t}$$

$$\leqslant \text{const}\,f^*(a)^{1/\alpha}\int_0^a \phi_{\bar{X}}(t)^{1/\alpha}\frac{d\phi_{\bar{X}}(t)}{\phi_{\bar{X}}(t)}$$

$$\leqslant \text{const}\,f^*(a)^{1/\alpha}\phi_{\bar{X}}(t)^{1/\alpha}\,\Big|_0^a < +\infty.$$

Corollary 3.2. *For* $0<\alpha\leqslant 1$, *we have*

$$\Lambda(X)\subset\Lambda_\alpha(X)\subset M(X)$$

with continuous embeddings.

Proof. From the definitions given for the extremal spaces Λ and M and from (2.1) we see that we need to show

$$\alpha^\alpha \phi_X(t) \leqslant \phi_{\Lambda_\alpha(X)}(t) \leqslant \text{const}\, \phi_X(t) \qquad t \in I, \tag{3.5}$$

since then $\Lambda(\Lambda_\alpha(X)) = \Lambda(X)$ and $M(\Lambda_\alpha(X)) = M(X)$ with equivalent norms. For the left-hand inequality notice

$$\begin{aligned}
\alpha^\alpha \phi_X(t) &= \alpha^\alpha \left\{ \int_0^l \chi_{(0,t)}(s)^{1/\alpha}\, d[\phi_X(s)^{1/\alpha}] \right\}^\alpha \\
&\leqslant \left\{ \int_0^l [\chi_{(0,t)}^{**}(s)\, \phi_X(s)]^{1/\alpha} \frac{d\phi_X(s)}{\phi_X(s)} \right\}^\alpha \\
&\leqslant \left\{ \int_0^l [\chi_{(0,t)}^{**}(s)\, \phi_X(s)]^{1/\alpha} \frac{ds}{s} \right\}^\alpha \\
&= \phi_{\Lambda_\alpha(X)}(t)
\end{aligned}$$

by (1.9) and the fact $\phi_X(0^+) = 0$.

Now we denote by $\bar{X}$ the space X equivalently renormed with properties of Theorem 2.4. By Lemma 3.1 and (2.9) we obtain

$$\begin{aligned}
\phi_{\Lambda_\alpha(X)}(t) &= \| \chi_{(0,t)} \|_{\Lambda_\alpha(X)} \leqslant \text{const}\, \| \chi_{(0,t)} \|_{\Lambda_\alpha(X)}^* \\
&= \text{const} \left\{ \int_0^t \phi_X(s)^{1/\alpha} \frac{ds}{s} \right\}^\alpha \leqslant \text{const} \left\{ \int_0^t \phi_{\bar{X}}(s)^{1/\alpha} \frac{d\phi_{\bar{X}}(s)}{\phi_{\bar{X}}(s)} \right\}^\alpha \\
&= \text{const}\, \phi_{\bar{X}}(t) \leqslant \text{const}\, \phi_X(t).
\end{aligned}$$

Corollary 3.3 *If $\beta < \alpha$, then $\Lambda_\alpha(X)$ is continuously embedded in $\Lambda_\beta(X)$. Moreover, $\Lambda_\alpha(X)$ is dense in $\Lambda_\beta(X)$.*

Proof. The last statement follows since simple functions are dense in $\Lambda_\gamma(X)$, $0 < \gamma \leqslant 1$. This assertion is proved as in the case of L^p spaces.

Now assume $\| f \|_{\Lambda_\alpha(X)} = 1$, then by Corollary 3.2

$$f^{**}(t)\, \phi_X(t) \leqslant \| f \|_{M(X)} \leqslant A \cdot \| f \|_{\Lambda_\alpha(X)} = A.$$

Since $1/\alpha < 1/\beta$

$$[f^{**}(t)\, \phi_X(t)/A]^{1/\beta} \leqslant [f^{**}(t)\, \phi_X(t)/A]^{1/\alpha},$$

so

$$\| f \|_{\Lambda_\beta(X)} \leqslant A^{1-\beta/\alpha} \| f \|_{\Lambda_\alpha(X)}^{\beta/\alpha} = A^{1-\beta/\alpha}.$$

PROPOSITION 3.4. *Suppose* $0 < \alpha < \beta < \gamma \leqslant 1$, *then for each* $f \in \Lambda_\gamma(X)$ *we have*

$$\|f\|_{\Lambda_\beta(X)} \leqslant \|f\|_{\Lambda_\alpha(X)}^{(\gamma-\beta)/(\gamma-\alpha)} \|f\|_{\Lambda_\gamma(X)}^{(\beta-\alpha)/(\gamma-\alpha)}.$$

Proof. This result follows from Corollary 3.3 and the logarithmically convex property of the norm of L^p spaces.

Remark 3.5. After a few preparatory lemmas we will be in a position to prove our duality result. The proof given here is very similar to that given by Oaklander [11] for L^{pq} spaces.

LEMMA 3.6. *Suppose* h *is a positive measurable function on* I *and satisfies* $\int_0^l [h(s)\,\phi_X(s)]^{1/\alpha}\,ds/s \leqslant 1$. *If* $f(t) = \int_t^l h(s)\,ds/s$, *then* $\|f\|_{\Lambda_\alpha(X)} \leqslant A$, *where* A *is independent of* h.

Proof. First note that f is a positive decreasing function on I. By the monotone-convergence theorem we can assume there exist $0 < a < b < l$ such that h is zero on $(0, a)$ and (b, l). Again we denote by $\bar{X}$ the space X equivalently renormed by Theorem 2.4. Applying Lemma 3.1 and statement (2.9), we obtain

$$\|f\|_{\Lambda_\alpha(X)}^{1/\alpha} \leqslant \text{const} \int_0^l [f^*(s)\,\phi_X(s)]^{1/\alpha} \frac{ds}{s} \leqslant \text{const} \int_0^l [f(s)\,\phi_{\bar{X}}(s)]^{1/\alpha} \frac{d\phi_{\bar{X}}(s)}{\phi_{\bar{X}}(s)}$$

$$= \text{const} \int_0^l f(s)^{1/\alpha}\, d[\phi_{\bar{X}}(s)^{1/\alpha}].$$

Since f is zero on (b, l), integrating by parts, we get

$$\|f\|_{\Lambda_\alpha(X)}^{1/\alpha} \leqslant \text{const} \left(- \int_0^l \phi_X(s)^{1/\alpha} f(s)^{1/\alpha-1}\, d[f(s)]\right).$$

But, $df(s)/ds = -h(s)/s$, so substituting and applying Hölder's inequality we have

$$\|f\|_{\Lambda_\alpha(X)}^{1/\alpha} \leqslant \text{const} \int_0^l f(s)^{1/\alpha-1}\, h(s)\, \phi_X(s)^{1/\alpha} \frac{ds}{s}$$

$$\leqslant \text{const} \left\{\int_0^l [f(s)\,\phi_X(s)]^{1/\alpha} \frac{ds}{s}\right\}^{1-\alpha} \left\{\int_0^l [h(s)\,\phi_X(s)]^{1/\alpha} \frac{ds}{s}\right\}^{\alpha}$$

$$\leqslant \text{const}\, \|f\|_{\Lambda_\alpha(X)}^{1/\alpha-1}.$$

But $\|f\|_{\Lambda_\alpha(X)} \leqslant \|f(a)\,\chi_{(0,b)}(t)\|_{\Lambda_\alpha(X)} \leqslant \text{const}\, f(a)\,\phi_X(b) < +\infty$, so

$$\|f\|_{\Lambda_\alpha(X)} \leqslant \text{const} = A.$$

LEMMA 3.7. *For each measurable function g on I and* $0 < \alpha < 1$

$$\| g \|_{\Lambda_{\alpha'}(X')} = \sup_{f \in A_\alpha} \int_0^l f^*(t)\, g^*(t)\, dt, \tag{3.6}$$

where $\alpha' = 1 - \alpha$ *and* $A_\alpha = \{f \mid f(t) = \int_t^l h(s)\, ds/s$, *where* $h \geqslant 0$ *and* $\int_0^l [h(s)\, \phi_X(s)]^{1/\alpha}\, ds/s = 1\}$.

Proof. Suppose $f \in A_\alpha$, then using Fubini and Hölder's inequality

$$\begin{aligned}
\int_0^l f(t)\, g^*(t)\, dt &= \int_0^l g^*(t) \int_t^l h(s) \frac{ds}{s}\, dt = \int_0^l \left(\int_0^s g^*(t)\, dt\right) h(s) \frac{ds}{s} \\
&= \int_0^l g^{**}(s)\, h(s)\, ds = \int_0^l [g^{**}(s)\, \phi_{X'}(s)][h(s)\, \phi_X(s)] \frac{ds}{s} \\
&\leqslant \left\{\int_0^l [g^{**}(s)\, \phi_{X'}(s)]^{1/\alpha'} \frac{ds}{s}\right\}^{\alpha'} \left\{\int_0^l [h(s)\, \phi_X(s)]^{1/\alpha} \frac{ds}{s}\right\}^{\alpha} \\
&= \| g \|_{\Lambda_{\alpha'}(X')}.
\end{aligned}$$

On the other hand, take M to be any number satisfying $0 \leqslant M < \| g \|_{\Lambda_{\alpha'}(X')}$. By the monotone-convergence theorem there exist numbers a, b between 0 and l such that

$$\left\{\int_a^b [g^{**}(s)\, \phi_{X'}(s)]^{1/\alpha'} \frac{ds}{s}\right\}^{\alpha'} > M. \tag{3.7}$$

We shall exhibit an $h \geqslant 0$ with the following properties:

(i) $f(t) = \int_t^l h(s)\, ds/s$ belongs to A_α;

(ii) $\int_0^l f^*(t)\, g^*(t)\, dt > M$.

Define

$$h(s) = [g^{**}(s)\, \phi_{X'}(s)/\beta]^{1/\alpha' - 1}\, \phi_X(s)^{-1}\, \chi_{(a,b)}(s),$$

where

$$\beta = \left\{\int_a^b [g^{**}(s)\, \phi_{X'}(s)]^{1/\alpha'} \frac{ds}{s}\right\}^{\alpha'}.$$

Since

$$\begin{aligned}
\int_0^l [h(s)\, \phi_X(s)]^{1/\alpha} \frac{ds}{s} &= \int_a^b \left[\frac{g^{**}(s)\, \phi_{X'}(s)}{\beta}\right]^{(1/\alpha' - 1)/\alpha} \frac{ds}{s} \\
&= \left\{\int_a^b [g^{**}(s)\, \phi_{X'}(s)]^{1/\alpha'} \frac{ds}{s}\right\} \beta^{-1/\alpha'} = 1,
\end{aligned}$$

we have that (i) is true.

Assertion (ii) is valid since

$$\int_0^l f^*(t)\,g^*(t)\,dt = \int_0^l g^{**}(t)\,h(t)\,dt = \beta^{1-1/\alpha'} \int_a^b g^{**}(t)^{1/\alpha'}\,\phi_{X'}(t)^{1/\alpha'}\,\frac{dt}{t}$$
$$= \beta^{1-1/\alpha'}\,\beta^{1/\alpha'} = \beta > M$$

by (3.7).

We need the following well-known result from the literature [10; Chapter I, Section 2, Theorem 3] and [5, p. 72];

PROPOSITION 3.8. *Suppose Z is a Banach function space. A necessary and sufficient condition that*

$$Z^* = Z'$$

is that $\|f\chi_E\|_Z$ converges to zero as mE goes to zero for each f in Z.

COROLLARY 3.9. *If $0 < \alpha < 1$, then*

$$\Lambda_\alpha(X)^* = \Lambda_\alpha(X)'.$$

Proof. By Lemma (3.1), we have

$$\|f\chi_E\|_{\Lambda_\alpha(X)} \leqslant \text{const}\,\|f\chi_E\|^*_{\Lambda_\alpha(X)} \leqslant \text{const}\left\{\int_0^l [f^*(t)\,\chi_{(0,mE)}(t)\,\phi_X(t)]^{1/\alpha}\,dt/t\right\}^\alpha.$$

But the last expression converges to 0 with mE by the dominated convergence theorem, so apply Proposition (3.8) to get the result.

THEOREM 3.10. *If $0 < \alpha < 1$, then, with equivalent norms,*

$$\Lambda_\alpha(X)^* = \Lambda_\alpha(X)' = \Lambda_{\alpha'}(X'),$$

where $\alpha' = 1 - \alpha$. In particular, $\Lambda_\alpha(X)$ is reflexive for $0 < \alpha < 1$.

Proof. In view of Corollary 3.9, we need only show $\Lambda_\alpha(X)' = \Lambda_{\alpha'}(X')$. Suppose $g \in \Lambda_{\alpha'}(X')$ and $\|f\|_{\Lambda_\alpha(X)} \leqslant 1$, then

$$\int_0^l f^*(t)\,g^*(t)\,dt \leqslant \int_0^l f^{**}(t)\,g^{**}(t)\,dt = \int_0^l [f^{**}(t)\,\phi_X(t)][g^{**}(t)\,\phi_{X'}(t)]\,\frac{dt}{t}.$$

Applying Hölder's inequality, we obtain

$$\int_0^l f^*(t)\,g^*(t)\,dt \leqslant \left\{\int_0^l [f^{**}(t)\,\phi_X(t)]^{1/\alpha}\,\frac{dt}{t}\right\}^\alpha \left\{\int_0^l [g^{**}(t)\,\phi_{X'}(t)]^{1/\alpha'}\,\frac{dt}{t}\right\}^{\alpha'}$$
$$\leqslant \|g\|_{\Lambda_{\alpha'}(X')}$$

Hence by the definition of the dual space,

$$\| g \|_{\Lambda_\alpha(X)'} \leqslant \| g \|_{\Lambda_{\alpha'}(X')} .$$

On the other hand, suppose $g \in \Lambda_\alpha(X)'$. For the norm $\| g \|_{\Lambda_{\alpha'}(X')}$, we have the formula (3.6). But by Lemma 3.6, if $f \in A_\alpha$, then $\| f \|_{\Lambda_\alpha(X)} \leqslant A$. Therefore

$$\| g \|_{\Lambda_{\alpha'}(X')} \leqslant \sup_{\|f\|_{\Lambda_\alpha(X)} \leqslant A} \int_0^l f^*(t)\, g^*(t)\, dt = A \| g \|_{\Lambda_\alpha(X)'} .$$

Remark 3.11. We sum up a few additional properties of the spaces $\Lambda_\alpha(X)$. For a fixed function f, $\| f \|_{\Lambda_\alpha(X)}$ is a continuous function of α. This follows from Proposition 3.4. If α tends to zero, then the norm of $\Lambda_\alpha(X)$ tends to the norm of $M(X)$ (same as $\| \|_{L^p} \to \| \|_{L^\infty}$ as $p \to \infty$); since the simples are dense in $\Lambda_\alpha(X)$ the "limit" of the spaces $\Lambda_\alpha(X)$ as $\alpha \to 0$ is the space $M^0(X)$, the norm closure of $L^1 \cap L^\infty$ in $M(X)$. By using Lemma 3.1 and Theorem 2.4 it is easy to show that $\Lambda_1(X) = \Lambda(X)$. Analogous to Theorem 3.10 is the previously known duality result

$$\Lambda(X)^* = \Lambda(X)' = M(X')$$

which is a special case of a theorem given in Lorentz [6, p. 73, Theorem 3.72] for $\Lambda_{\psi,1/\alpha}$. Also by Theorem 3 in Chapter 1, Section 2 of [10] we have

$$M^0(X)^* = \Lambda(X).$$

4. Calderón's Theorem

We begin by stating a theorem due to Calderón [3]. Our aim is to extend this theorem to a function-space setting without the restrictions of Zippin which were outlined in Section 1. This is our Theorem 4.7.

Let σ be the closed line segment in the unit square connecting the points (α_1, β_1) and (α_2, β_2), where $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$. Define the operator $S(\sigma)$ acting on f pointwise by

$$S(\sigma)[f](t) = \int_0^l f(s)\, d\Psi(s, t),$$

where

$$\Psi(s, t) = \min_{i=1,2} \left\{ \frac{s^{\alpha_i}}{t^{\beta_i}} \right\}.$$

We can now state the theorem of Calderón:

THEOREM A (See [3, Theorem 10]). *In order that each operator mapping Λ_{α_i} boundedly into $M_{1-\beta_i}$, $i = 1, 2$, should have a unique extension to a bounded operator from X to Y, it is necessary and sufficient that $S(\sigma)$ should map X to Y.*

We assume that all spaces are renormed to possess concave fundamental functions, and we let $\sigma = [(X_1, Y_1), (X_2, Y_2)]$. Instead of the condition $\alpha_1 \neq \alpha_2$, we assume the weaker condition

$$\min_{i=1,2} \phi_{X_i}(0^+) = 0 \tag{4.1}$$

holds. This gives us a minimal separation of the first coordinate of the interpolation segment σ; indeed, if $l < \infty$, then at least one of the spaces $\Lambda(X_i)$ is not equal to L^∞.

We now define Calderón's operator $S(\sigma)$ for the general situation (see [16]). Define

$$\Psi(s, t) = \min_{i=1,2} \left\{ \frac{\phi_{X_i}(s)}{\phi_{Y_i}(t)} \right\} \quad \text{for } s, t \in I. \tag{4.2}$$

For fixed t, Ψ is a continuous increasing concave function of s (the minimum of two concave functions is concave). By (4.1) we have $\lim_{s\to 0} \Psi(s, t) = 0$. Hence the function

$$\Phi(s, t) = d\Psi(s, t)/ds \tag{4.3}$$

is a positive decreasing function on I and

$$\int_0^r \Phi(s, t)\, ds = \Psi(r, t). \tag{4.4}$$

The operator $S(\sigma)$ is defined for each f pointwise by

$$S(\sigma)[f](t) = \int_0^l f(s)\, \Phi(s, t)\, ds \tag{4.5}$$

if the integral is absolutely convergent for almost all t. The following two lemmas are available in the literature (see [1, 3], and [16]).

LEMMA 4.1. *Suppose $f \in \Lambda(X_1) + \Lambda(X_2)$, then*

$$S(\sigma)[f^*] \text{ is a positive decreasing function which is finite valued on } I, \tag{4.6}$$

$$S(\sigma)[f] \text{ is defined,} \tag{4.7}$$

$$(S(\sigma)[f])^*(t) \leqslant S(\sigma)[f^*](t), \quad \text{all } t. \tag{4.8}$$

LEMMA 4.2. *The operator $S(\sigma)$ is of weak types (X_i, Y_i), for $i = 1, 2$, where $\sigma = [(X_1, Y_1), (X_2, Y_2)]$. In fact, for $i = 1, 2$, $\| S(\sigma)\|_{w(X_i,Y_i)} = 1$.*

The decomposition result which follows is crucial for the proof of our generalization of Calderón's theorem.

LEMMA 4.3. *Let $\mathscr{D}$ denote the set of all positive decreasing functions on I. If $f \in \mathscr{D}$ then for each t there exist functions f_1 and f_2 in $\mathscr{D}$ such that $f_1 + f_2 = f$ and*

$$\int_0^l f_i(s)\, \Phi(s, t)\, ds = \frac{\| f_i \|_{\Lambda(X_i)}}{\phi_{Y_i}(t)} \qquad i = 1, 2. \tag{4.9}$$

Proof. Assume t is fixed. Let E be the subset of I such that $s \in E$ implies

$$\phi_{X_1}(s)/\phi_{Y_1}(t) < \phi_{X_2}(s)/\phi_{Y_2}(t). \tag{4.10}$$

There exists a countable disjoint union of open intervals $I_j = (a_j, b_j)$ such that $\bigcup_{j=1}^{\infty} I_j = E$. Note that we have equality in (4.10) if $s = a_j$ or b_j.

We now since f into horizontal pieces as follows: Let $g_j(s) = [\min(f(s), f(a_j)) - f(b_j)]_+$. The functions $g_j \in \mathscr{D}$ and satisfy

$$\begin{aligned}\int_0^l g_j(s)\, \Phi(s, t)\, ds &= \int_0^l g_j(s)\, d\Psi(s, t)\\ &= g_j(0^+)\frac{\phi_{X_1}(a_j)}{\phi_{Y_1}(t)} + \Big(\int_{I_j} g_j(s)\, d\phi_{X_1}(s)\Big)\Big/\phi_{Y_1}(t)\\ &= \Big(\int_0^l g_j(s)\, d\phi_{X_1}(s)\Big)\Big/\phi_{Y_1}(t) = \| g_j \|_{\Lambda(X_1)}/\phi_{Y_1}(t).\end{aligned}$$

Now let $f_1 = \sum_{j=1}^{\infty} g_j$, then $f_1 \in \mathscr{D}$ and

$$\begin{aligned}\int_0^l f_1(s)\, \Phi(s, t)\, ds &= \sum_j \int_0^l g_j(s)\, \Phi(s, t)\, ds = \sum_j \| g_j \|_{\Lambda(X_1)}/\phi_{Y_1}(t)\\ &= \| f_1 \|_{\Lambda(X_1)}/\phi_{Y_1}(t),\end{aligned}$$

since the norm of a Λ space is covariantly additive.

For the function f_2 consider $f - f_1$. Since $f_2 = \lim_{n\to\infty} (f - \sum_{j=1}^{n} g_j)$, we see that $f_2 \in \mathscr{D}$ and is constant on each I_j. Let $E' = I \setminus E$ and notice that $\Phi(s, t) = (d\phi_{X_2}(s)/ds)/\phi_{Y_2}(t)$ almost everywhere on E'. The exceptional set is the set of isolated points of E' or where $d\phi_{X_2}(s)/ds$ or $d\Psi(s, t)/ds$ does not exist. So,

$$\int_0^l f_2(s)\, \Phi(s, t)\, ds = \left(\int_E + \int_{E'}\right) f_2(s)\, \Phi(s, t)\, ds$$

$$= \sum_j \int_{I_j} f_2(s)\, \Phi(s, t)\, ds + \left(\int_{E'} f_2(s)\, d\phi_{X_2}(s)\right)\Big/\phi_{Y_2}(t).$$

But f_2 is constant on each I_j, so by equality in (4.10)

$$\int_0^l f_2(s)\, \Phi(s, t)\, ds = \left(\int_E f_2(s)\, d\phi_{X_2}(s)\right)\Big/\phi_{Y_2}(t) + \left(\int_{E'} f_2(s)\, d\phi_{X_2}(s)\right)\Big/\phi_{Y_2}(t)$$

$$= \| f_2 \|_{\Lambda(X_2)}/\phi_{Y_2}(t).$$

COROLLARY 4.4 *If for some* $t_0 \in I$, $\int_0^l f^*(s)\, \Phi(s, t_0)\, ds < +\infty$, *then* $f \in \Lambda(X_1) + \Lambda(X_2)$.

Proof. We can assume that f is a positive decreasing function since $\Lambda(X_1) + \Lambda(X_2)$ is a rearrangement-invariant space. Applying Lemma 4.3, we can find functions $f_1, f_2 \in \mathscr{D}$ such that $f_1 + f_2 = f$ and (4.9) holds. Hence

$$c \| f \|_{\Lambda(X_1)+\Lambda(X_2)} = c \inf_{g_1+g_2=f} \{\| g_1 \|_{\Lambda(X_1)} + \| g_2 \|_{\Lambda(X_2)}\}$$

$$\leqslant \| f_1 \|_{\Lambda(X_1)}/\phi_{Y_1}(t_0) + \| f_2 \|_{\Lambda(X_2)}/\phi_{Y_2}(t_0)$$

$$= \int_0^l f_1(s)\, \Phi(s, t_0)\, ds + \int_0^l f_2(s)\, \Phi(s, t_0)\, ds$$

$$= \int_0^l f(s)\, \Phi(s, t_0)\, ds < +\infty,$$

where $c = \min_{i=1,2} \{\phi_{Y_i}(t_0)^{-1}\}$.

THEOREM 4.5. *Suppose T is an operator of weak types (X_i, Y_i), $i = 1, 2$, then for each $f \in \Lambda(X_1) + \Lambda(X_2)$,*

$$(Tf)^*(t) \leqslant 2 \cdot (\max_{i=1,2} \| T \|_{w(X_i, Y_i)})\, S(\sigma)[f^*](t). \tag{4.11}$$

Proof. Suppose $t \in I$. By Lemma 4.3 there exist positive decreasing functions f_1 and f_2 such that $f_1 + f_2 = f^*$ and (4.9) holds. Under the measure-preserving map $f^* \to f$ (*see* [3, Lemma 2), we obtain functions

g_1, g_2 such that $g_i^* = f_i$ and $g_1 + g_2 = f$, a.e. Since T is of weak types (X_i, Y_i), we have

$$\begin{aligned}(Tg_i)^*(t/2) &\leqslant \| T \|_{w(X_i,Y_i)} \| g_i \|_{\Lambda(X_i)} / \phi_{Y_i}(t/2) \\ &= \| T \|_{w(X_i,Y_i)} \| f_i \|_{\Lambda(X_i)} / \phi_{Y_i}(t/2).\end{aligned}$$

But ϕ_{Y_i} is concave, so $\phi_{Y_i}(t) \leqslant 2\phi_{Y_i}(t/2)$, and

$$\begin{aligned}(Tg_i)^*(t/2) &\leqslant 2 \cdot \| T \|_{w(X_i,Y_i)} \cdot \| f_i \|_{\Lambda(X_i)} / \phi_{Y_i}(t) \\ &= 2 \| T \|_{w(X_i,Y_i)} \cdot S(\sigma)[f_i](t).\end{aligned}$$

We have the following inequality from [3, p. 290, line 7–]:

$$(Tf)^*(t) \leqslant (Tg_1)^*(t/2) + (Tg_2)^*(t/2)$$

so

$$\begin{aligned}(Tf)^*(t) &\leqslant 2(\max_{i=1,2} \| T \|_{w(X_i,Y_i)})(S(\sigma)[f_1](t) + S(\sigma)[f_2](t)) \\ &= 2(\max_{i=1,2} \| T \|_{w(X_i,Y_i)})\, S(\sigma)[f^*](t).\end{aligned}$$

Remark 4.6. Property (4.11) gives justification for the use of the terminology "maximal weak-type (X_i, Y_i) operator" in referring to the operator $S(\sigma)$.

THEOREM 4.7. *A necessary and sufficient condition for (X, Y) to be weak intermediate for σ is that for each $f \in X$, $S(\sigma)[f]$ should exist, and $S(\sigma)[f] \in Y$.*

Proof. The necessity of the condition is obvious since $S(\sigma)$ is a weak-type (X_i, Y_i) operator and hence must map X to Y.

On the other hand, let us show that the condition is sufficient. Since $S(\sigma) f \in Y$, $S(\sigma) f$ must be finite almost everywhere. By Corollary 4.4, $f \in \Lambda(X_1) + \Lambda(X_2)$. Hence $X \subset \Lambda(X_1) + \Lambda(X_2)$; this embedding is continuous by the closed-graph theorem. This shows that X is contained in the domain of any weak-type (X_i, Y_i) operator, $i = 1, 2$, so the extension of operators is trivial. We need to show that $S(\sigma)$ is a bounded map from X to Y. Again we employ the closed-graph theorem. Suppose $f_n \to f$ in X and $S(\sigma)[f_n] \to g$ in Y, then $f_n \to f$ in $\Lambda(X_1) + \Lambda(X_2)$ by the continuous embedding. Since $S(\sigma)$ is of weak types (X_i, Y_i), $i = 1, 2$, we have that $S(\sigma)[f_n] \to S(\sigma)[f]$ in measure on subsets of I of finite measure. Since Y is a rearrangement-invariant space $S(\sigma)[f_n] \to g$ in measure on sets of finite measure. Hence $g = S(\sigma)[f]$.

We now apply Theorem 4.5.

Remark 4.8. In [3] and [16] no attempt was made to find the best constant C such that

$$\| Tf \|_Y \leqslant C \| f \|_X ,$$

where $\max_{i=1,2} \| T \|_{w(X_i, Y_i)} \leqslant 1$, and $f \in X$. Theorem 4.5 and Lemma 4.2 do not give us the best constant C, but they do yield

$$\| S(\sigma) \|_{X \to Y} \leqslant C \leqslant 2 \| S(\sigma) \|_{X \to Y} .$$

In the classical case $\sigma = [(L^{p_1}, L^{q_1}), (L^{p_2}, L^{q_2})]$ we get that

$$\| S(\sigma) \|_{L^{pr} \to L^{qr}} \approx 1/\theta(1 - \theta),$$

where $1/p = \theta/p_1 + (1 - \theta)/p_2$ and $1/q = \theta/q_1 + (1 - \theta)/q_2$.

We now interpret Theorem 4.7 into the language of strong interpolation theorems.

THEOREM 4.9. *Suppose* $M_{\phi_i} \in \mathscr{U}$, $i = 1, 2$, *and let* $\sigma = [(\Lambda_{\phi_1}, M_{\psi_1}), (\Lambda_{\phi_2} M_{\psi_2})]$. *The pair* (X, Y) *is trong intermediate for* σ *if and only if for each* $f \in X$, $S(\sigma) f$ *exists and belongs to* Y.

5. INTERPOLATION THEOREMS

We assume X_1, X_2, Y_1, Y_2, X, and Y have been equivalently renormed to possess concave fundamental functions and that (4.1) holds.

We give necessary and sufficient conditions in terms of the function

$$F(s, t) = \Psi(s, t) \cdot \phi_Y(t)/\phi_X(s) \qquad s, t \in I \tag{5.1}$$

for special spaces, say Λ, M, and $\Lambda_\alpha(X)$, to be weak intermediate for an interpolation segment σ. Computation involving the function F also yields a sufficient condition for (X, Y) to be strong intermediate for σ and a necessary condition in order that (X, Y) be weak intermediate for σ. These results depend heavily upon the results obtained in Section 4.

THEOREM 5.1. *A necessary and sufficient condition that* $(\Lambda(X), \Lambda(Y))$ *be weak intermediate for* σ *is*

$$\int_0^l F(s, t) \frac{d\phi_Y(t)}{\phi_Y(t)} \leqslant A \qquad s \in I. \tag{5.2}$$

Proof. The condition is shown to be necessary by considering simple functions of the form $f_s(t) = \chi_{(0,s)}(t)$. We assume that $(\Lambda(X), \Lambda(Y))$ is weak intermediate for σ. Then by Lemma 4.2.

$$\| S(\sigma) f_s \|_{\Lambda(Y)} \leqslant A \| f_s \|_{\Lambda(X)} .$$

But by (4.4)

$$S(\sigma) f_s(t) = \Psi(s, t) \qquad t \in I,$$

so

$$\int_0^l F(s, t) \frac{d\phi_Y(t)}{\phi_Y(t)} = \Big[\int_0^l \Psi(s, t)\, d\phi_Y(t)\Big] \Big/ \phi_X(s) = \frac{\| S(\sigma) f_s \|_{\Lambda(Y)}}{\phi_X(s)}$$
$$\leqslant A \| f_s \|_{\Lambda(X)} / \phi_X(s) = A.$$

On the other hand, assume (5.2) holds and let h be a positive decreasing simple function, say

$$h(t) = \sum_{i=1}^n a_i \chi_{(0,s_i)}(t),$$

where $s_i \in I$ and $a_i \geqslant 0$. By the covariantly additive property of the norm in $\Lambda(X)$ we obtain

$$\begin{aligned}
\| S(\sigma) h \|_{\Lambda(Y)} &\leqslant \sum_i a_i \| S(\sigma)\, \chi_{(0,s_i)} \|_{\Lambda(Y)} \\
&= \sum_i a_i \Big\{ \int_0^l F(s_i, t)/\phi_Y(t)\, d\phi_Y(t) \Big\} \phi_X(s_i) \\
&\leqslant A \sum_i a_i \phi_X(s_i) = A \sum_i \int_0^l a_i \chi_{(0,s_i)}(t)\, d\phi_X(t) \\
&= A \| h \|_{\Lambda(X)} .
\end{aligned}$$

Now let $f \in \Lambda(X)$ and let $\{h_n\}$ be a monotone increasing sequence of positive decreasing simple functions converging to f^*. By the monotone-convergence theorem $S(\sigma)[h_n](t)$ converges up to $S(\sigma)[f^*](t)$. By Lemma 4.1 and the definition of rearrangement-invariant spaces we have

$$\| S(\sigma) f \|_{\Lambda(Y)} \leqslant \| S(\sigma)[f^*] \|_{\Lambda(Y)} \leqslant \lim_{n\to\infty} \| S(\sigma)[h_n] \|_{\Lambda(Y)}$$
$$\leqslant A \lim_{n\to\infty} \| h_n \|_{\Lambda(X)} \leqslant A \| f \|_{\Lambda(X)} .$$

We finish by applying Theorem 4.7.

THEOREM 5.2. *Assume X and $Y \in \mathscr{U}$. A necessary and sufficient condition that $(M(X), M(Y))$ is weak intermediate for σ is*

$$\int_0^l F(s,t)\,\frac{d\phi_X(s)}{\phi_X(s)} \leqslant A \qquad t \in I. \tag{5.3}$$

Proof. First we make the following observation

$$\int_0^l F(s,t)\,\frac{d\phi_X(s)}{\phi_X(s)} = S(\sigma)[g](t)\cdot\phi_Y(t), \tag{5.4}$$

where $g(s) = 1/\phi_X(s)$. To show this, let a_n, $b_n \in I$ such that $a_n \downarrow 0$ and $b_n \uparrow l$. If we define

$$g_n(s) = \min\{g(a_n), g(s)\,\chi_{(0,b_n)}(s)\}$$

then

$$\int_{a_n}^{b_n} F(s,t)\,\frac{d\phi_X(s)}{\phi_X(s)} = \phi_Y(t)\int_{a_n}^{b_n} \Psi(s,t)\,d\left[\frac{-1}{\phi_X(s)}\right]$$

$$= -\phi_Y(t)\int_0^l \Psi(s,t)\,d[g_n(s)] = \phi_Y(t)\,S(\sigma)[g_n](t)$$

using integration by parts since $\Psi(0+,t) = 0$. Now we use the monotone-convergence theorem to get (5.4) since $g_n \uparrow g$.

Since $X \in \mathscr{U}$, by Theorem 2.2 we get

$$\sup_t\{f^*(t)\,\phi_X(t)\} \leqslant \|f\|_{M(X)} \leqslant C(X)\sup_t\{f^*(t)\,\phi_X(t)\}.$$

A similar statement holds for Y. Hence if $\|f\|_{M(X)} \leqslant 1$, then $f^*(t) \leqslant g(t)$ and

$$\|S(\sigma)f\|_{M(Y)} \leqslant \|S(\sigma)f^*\|_{M(Y)} \leqslant \|S(\sigma)g\|_{M(Y)}$$

$$\leqslant C(Y)\sup_t\{S(\sigma)[g](t)\,\phi_Y(t)\} \leqslant C(Y)\cdot A$$

by (5.4) and (5.3).

Condition (5.3) is necessary since $\|g\|_{M(X)} \leqslant C(X)$ and so

$$\int_0^l F(s,t)\,\frac{d\phi_X(s)}{\phi_X(s)} = S(\sigma)[g](t)\cdot\phi_Y(t) \leqslant \|S(\sigma)[g]\|_{M(Y)}$$

$$\leqslant \|S(\sigma)\|\,\|g\|_{M(X)} \leqslant C(X)\,\|S(\sigma)\| = A.$$

Remark 5.3. Theorems 5.1 and 5.2 give interpolation theorems with Lorentz Λ_ϕ and M_ϕ spaces since $\Lambda_\phi = \Lambda(\Lambda_\phi)$ and $M_\phi = M(M_\phi)$.

For the remainder of this section we require that all spaces belong to $\mathscr{L} \cap \mathscr{U}$. In this case we have that

$$\int_0^l \frac{F(s, t)\, dt}{t} \leqslant A \qquad s \in I \tag{5.5}$$

and

$$\int_0^l \frac{F(s, t)\, ds}{s} \leqslant A \qquad t \in I \tag{5.6}$$

are equivalent to conditions (5.2) and (5.3), respectively, by Theorem 2.4.

DEFINITION 5.4. We define the σ-finite measure μ on I by

$$\mu(E) = \int_E \frac{dt}{t},$$

where E is any Lebesgue measurable subset of I. If g is a measurable function on I such that $\mu\{s \mid |g(s)| > \epsilon\} < +\infty$ for each positive ϵ, then we denote by $g^{\circledast}$ the unique right-continuous positive decreasing function defined on I which is equimeasurable with g in the sense that

$$m\{s \mid g^{\circledast}(s) > \epsilon\} = \mu\{s \mid |g(s)| > \epsilon\}$$

for each positive ϵ. For each $a > 0$, it is clear that there exists a measurable subset E_a of I such that $\mu(E_a) = a$,

$$\int_0^a g^{\circledast}(s)\, ds = \int_{E_a} \frac{|g(s)|\, ds}{s}, \tag{5.7}$$

and

$$|g(s)| \leqslant g^{\circledast}(a) \quad s \in I \setminus E_a \quad \text{a.e.} \tag{5.8}$$

Properties (1.1) and (1.2), along with two minor properties, can be used to define Banach function spaces over the measure space (I, μ) Property (1.4) is, in fact, proved in [10] for Banach function spaces over arbitrary σ-finite measure spaces. Using (1.4), we can easily see that

$$\|f\|_{Z(\mu)} = \sup_{\|g\|_{Z'(\mu)} \leqslant 1} \left| \int_0^l \frac{f(s)\, g(s)\, ds}{s} \right| = \sup_{\|g\|_{Z'(\mu)} \leqslant 1} \int_0^\infty f^{\circledast}(s)\, g^{\circledast}(s)\, ds \tag{5.9}$$

holds for all rearrangement-invariant spaces $Z(\mu)$ over (I, μ). As in the case for (I, m) we have

$$\text{if } h^{\circledast} \prec f^{\circledast}, \text{ then } \|h\|_{Z(\mu)} \leqslant \|f\|_{Z(\mu)} \tag{5.10}$$

by property (5.9). A rearrangement invariant space over (I, μ) of special interest is the space $L^p(\mu)$ used in the discussion below. For the details of this summary, see Butzer–Berens [2].

LEMMA 5.5. *Assume conditions* (5.5) *and* (5.6) *are satisfied, then if* $(f^*\phi_X)^\circledast$ *exists,* $(S(\sigma)[f^*]\,\phi_Y)^\circledast$ *exists and*

$$(S(\sigma)[f^*]\,\phi_Y)^\circledast \prec A(f^*\phi_X)^\circledast. \tag{5.11}$$

Proof. We define an operator T for positive functions f as follows

$$Tf(t) = \int_0^l \frac{F(s, t)\, f(s)\, ds}{s} \qquad t \in I.$$

Since $d\phi_{X_i}(s)/ds \leqslant \phi_{X_i}(s)/s$ a.e., we have

$$\begin{aligned} S(\sigma)\left[\frac{f}{\phi_X}\right](t)\,\phi_Y(t) &= \phi_Y(t) \int_0^l \frac{f(s)}{\phi_X(s)}\, \Phi(s, t)\, ds \\ &\leqslant \phi_Y(t) \int_0^l \frac{f(s)}{\phi_X(s)} \frac{1}{s}\, \Psi(s, t)\, ds \\ &= Tf(t). \end{aligned}$$

If we can show

$$(Tf)^\circledast \prec Af^\circledast \tag{5.12}$$

for each function f, in particular for $f^*\phi_X$, then

$$(S(\sigma)[f^*]\,\phi_Y)^\circledast \leqslant (T[f^*\phi_X])^\circledast \prec A(f^*\phi_X)^\circledast$$

and we will be done.

By (5.7) we can find sets E_1 and E_2 for a given $a > 0$ such that $\mu(E_i) = a$ and

$$\int_{E_1} \frac{|\,Tf(t)|\, dt}{t} = \int_0^a (Tf)^\circledast(t)\, dt, \qquad \int_{E_2} \frac{|f(s)|\, ds}{s} = \int_0^a f^\circledast(s)\, ds.$$

Now let $g(s) = \int_{E_1} F(s, t)\, dt/t$, $s \in I$. By (5.5) we have $\sup_{s\in I} g(s) \leqslant A$, while (5.6) implies $\int_0^l g(s)\, ds/s \leqslant aA$. Hence

$$\begin{aligned} \int_0^a (Tf)^\circledast(t)\, dt = \int_{E_1} \frac{|\,Tf(t)|\, dt}{t} &\leqslant \int_{E_1} \frac{dt}{t} \int_0^l \frac{F(s, t)\,|f(s)|\, ds}{s} \\ &= \int_0^l \frac{|f(s)|\, ds}{s} \int_{E_1} \frac{F(s, t)\, dt}{t} = \int_0^l \frac{|f(s)|\, g(s)\, ds}{s} \end{aligned} \tag{5.13}$$

by the theorem of Fubini.

By (5.8) we also have

$$\begin{aligned}\int_0^l \frac{|f(s)|\, g(s)\, ds}{s} &= \int_{E_2} \frac{|f(s)|\, g(s)\, ds}{s} + \int_{I\setminus E_2} \frac{|f(s)|\, g(s)\, ds}{s} \\ &\leqslant \int_{E_2} \frac{|f(s)|\, g(s)\, ds}{s} + f^{\circledast}(a) \int_{I\setminus E_2} \frac{g(s)\, ds}{s} \\ &= \int_{E_2} \frac{[|f(s)| - f^{\circledast}(a)]\, g(s)\, ds}{s} + f^{\circledast}(a) \int_0^l \frac{g(s)\, ds}{s} \\ &\leqslant A \int_{E_2} \frac{[|f(s)| - f^{\circledast}(a)]\, ds}{s} + Aaf^{\circledast}(a) \\ &= A \int_{E_2} \frac{|f(s)|\, ds}{s} = A \int_0^a f^{\circledast}(s)\, ds.\end{aligned}$$

Combining this with inequality (5.13) we obtain the desired result.

THEOREM 5.6. *We let $\Lambda_0(X) = M(X)$, then for each $0 \leqslant \alpha \leqslant 1$ $(\Lambda_\alpha(X), \Lambda_\alpha(Y))$ is weak intermediate for σ if conditions* (5.5) *and* (5.6) *are satisfied. Moreover,* (5.5) *holds if $(\Lambda_1(X), \Lambda_1(Y))$ is weak intermediate for σ, while $(\Lambda_0(X), \Lambda_0(Y))$ being weak intermediate for σ implies* (5.6) *is true.*

Proof. The second part of the theorem follows from Theorems 5.1 and 5.2 and Remark 5.3. Theorems 5.1 and 5.2 also imply the first part of the theorem is true for $(\Lambda_1(X), \Lambda_1(Y))$ and $(\Lambda_0(X), \Lambda_0(Y))$.

Now suppose $0 < \alpha < 1$ and let $p = 1/\alpha$. Consider the rearrangement-invariant space (with respect to μ on I) $Z = L^p(\mu)$. Since (5.11) holds, then (5.10) implies

$$\| S(\sigma)[f^*]\, \phi_Y \|_{L^p(\mu)} \leqslant A \| f^* \phi_X \|_{L^p(\mu)}\,. \tag{5.14}$$

But (4.8) and (3.2) imply

$$\begin{aligned}\| S(\sigma) f \|_{\Lambda_\alpha(Y)} &\leqslant \| S(\sigma) f^* \|_{\Lambda_\alpha(Y)} \leqslant C \| S(\sigma) f^* \|^*_{\Lambda_\alpha(Y)} \\ &= C \| S(\sigma)[f^*]\, \phi_Y \|_{L^p(\mu)}\,.\end{aligned} \tag{5.15}$$

Again employing (3.2), we have

$$\| f^* \phi_X \|_{L^p(\mu)} = \| f \|^*_{\Lambda_\alpha(X)} \leqslant \| f \|_{\Lambda_\alpha(X)}\,. \tag{5.16}$$

Combining inequalities (5.15), (5.14), and (5.16), we obtain

$$\| S(\sigma) f \|_{\Lambda_\alpha(Y)} \leqslant AC \| f \|_{\Lambda_\alpha(X)}\,.$$

We are done by Theorem 4.7.

THEOREM 5.7. *The condition*

$$F(s, t) \leqslant A, \qquad s, t \in I \tag{5.17}$$

is necessary in order that (X, Y) *be weak intermediate for* σ.

Proof. Suppose (X, Y) is weak intermediate for σ, then $S(\sigma)$ maps X boundedly into Y. But (2.1) gives that $S(\sigma)$ is a bounded map from $\Lambda(X)$ into $M(Y)$. Hence if $f_s = \chi_{(0,s)}$ we have

$$\begin{aligned} F(s, t)\,\phi_X(s) &\leqslant S(\sigma) f_s(t)\,\phi_Y(t) \leqslant \| S(\sigma) f_s \|_{M(Y)} \\ &\leqslant A \| f_s \|_{\Lambda(X)} = A\phi_X(s) \end{aligned}$$

for each $s, t \in I$. We then have

$$F(s, t) \leqslant A.$$

Remark 5.8. The proof above indicates that (5.17) is necessary and sufficient for each operator of weak types (X_i, Y_i), $i = 1, 2$, to be an operator of weak-type (X, Y) (covariant additivity of the norm of $\Lambda(X)$).

Note that the proof, using (2.1), makes use of the fact that each strong operator mapping X into Y is of weak type (X, Y).

THEOREM 5.9.

$$\int_0^l \int_0^l F(s, t) \frac{ds}{s} \frac{dt}{t} \leqslant A \tag{5.18}$$

is sufficient for (X, Y) *to be weak (and hence strong) intermediate for* σ.

Proof. As noted in Remark 5.8 each operator T which maps X_i boundedly to Y_i is also a weak (X_i, Y_i) operator. If (X, Y) can be shown to be weak intermediate for σ, then T will map X boundedly to Y. Hence (X, Y) is strong intermediate for σ.

If we show that (5.18) implies $S(\sigma)$ maps $M(X)$ boundedly into $\Lambda(Y)$, then by the embedding (2.1) we have that $S(\sigma)$ maps X boundedly into Y. We proceed as in Theorem 5.2. Let $g(s) = 1/\phi_X(s)$. Suppose $f \in M(X)$ and $\|f\|_{M(X)} \leqslant 1$, then $f^*(t) \leqslant 1/\phi_X(t) = g(t)$. By (5.4) we have

$$\begin{aligned} (S(\sigma)f)^*(t) &\leqslant S(\sigma)[f^*](t) \leqslant S(\sigma)[g](t) \\ &= \frac{1}{\phi_Y(t)} \int_0^l F(s, t) \frac{d\phi_X(s)}{\phi_X(s)}. \end{aligned}$$

So

$$\| S(\sigma)f \|_{\Lambda(Y)} \leqslant \int_0^l \int_0^l \frac{F(s,t)\, d\phi_X(s)}{\phi_X(s)} \frac{d\phi_Y(t)}{\phi_Y(t)}$$
$$\leqslant \int_0^l \int_0^l \frac{F(s,t)\, ds}{s} \frac{dt}{t} \leqslant A$$

for each $\| f \|_{M(X)} \leqslant 1$.

From the preceding proof it is easy to see that condition (5.18) is also necessary in order that $(M(X), \Lambda(Y))$ be weak intermediate for σ.

Remark 5.10. In [1] Boyd gives a necessary and sufficient condition for (X, X) to be weak intermediate for $\sigma = [(L^{p_1}, L^{p_1}), (L^{p_2}, L^{p_2})]$. The condition is

$$1/p_2 < \underline{\sigma}_X \leqslant \bar{\sigma}_X < 1/p_1 ,$$

where $\underline{\sigma}_X$ and $\bar{\sigma}_X$ are the lower and upper indices of X calculated from the norm of the compression operator on the space X.

Zippin proves in [16] that if L^∞ is dense in all spaces under consideration, then the condition

$$\bar{\gamma}_{X_2} < \underline{\gamma}_X \leqslant \bar{\gamma}_X < \underline{\gamma}_{X_1} \tag{5.19}$$

is sufficient for (X, X) to be weak intermediate for

$$\sigma = [(X_1, X_1), (X_2, X_2)],$$

where the indices γ are discussed in Remark 1.4. It is not difficult to see that (5.19) implies both (5.2) and (5.3). On the other hand Shimogaki [14] gives an example of a rearrangement-invariant space X which satisfies

$$1/p_2 < \underline{\gamma}_X < \bar{\gamma}_X < 1/p_1 ,$$

but (X, X) is not weak intermediate for $\sigma = [(L^{p_1}, L^{p_1}), (L^{p_2}, L^{p_2})]$. Hence even though (5.2) and (5.3) are sufficient to give many pairs (see Lemma 5.5) as weak intermediate, they do not give them all. If in our theorems we let $X_i = Y_i$ and $X = Y$, then our results are not comparable with those of Zippin [16].

6. Comparison to Other Interpolation Methods

We compare the methods of Calderón with those of Lorentz–Shimogaki and Peetre for the interpolation segment

$$\bar{\sigma} = [(\Lambda(X_1), M^*(Y_1)), (\Lambda(X_2), M^*(Y_2))].$$

In both cases the methods are shown to be equivalent up to constants.

For a rearrangement-invariant space X we show that $\{\Lambda_\alpha(X)\}_{0\leqslant\alpha\leqslant 1}$ forms a scale in the sense of [5]. We then compare the theorems following from this fact to those obtained in Section 5.

We define $M^*(Y) = \{g \mid \|g\|_{M^*(Y)} = \sup_t\ g^*(t)\,\phi_Y(t) < +\infty\}$. Notice that $M(Y)$ is continuously embedded in $M^*(Y)$ since $g^*(t) \leqslant g^{**}(t)$, but $M^*(Y)$ is not a Banach space since $\|\cdot\|_{M^*(Y)}$ is not subadditive. $M^*(Y)$, however, is a complete quasilinear rearrangement-invariant function space satisfying properties (1.1) and (1.2). In most cases, i.e., $Y \in \mathscr{U}$, $M^*(Y) = M(Y)$ with equivalent norms by Theorem 2.2. The space $M^*(Y)$ has the property that bounded operators from $\Lambda(X)$ to $M^*(Y)$ are equivalent to weak-type (X, Y) operators.

DEFINITION 6.1. (see [8]). Suppose σ is an interpolation segment such that (4.1) holds, then we define the quasiorder $\prec$ with respect to σ by

> if $g \in Y_1 + Y_2$ and $f \in X_1 + X_2$, then $g \prec f$ means the following: Given any decomposition of f, say $f_1 + f_2 = f$, where $f_i \in X_i$, there is a decomposition of g, say $g_1 + g_2 = g$, where
>
> $$g_i \in Y_i \qquad \text{and} \qquad \|g_i\|_{Y_i} \leqslant \|f_i\|_{X_i}, \qquad i = 1, 2.$$

We define $\mathscr{L}\mathscr{S}(\sigma)$ to be the class of all pairs (X, Y) which satisfy the condition: There is a constant C such that

$$g \prec f \text{ with respect to } \sigma \text{ implies } \|g\|_Y \leqslant C\|f\|_X .$$

The elementary proof of the following result also works for the spaces M^*, so we extend the definition of $\prec$ to quasilinear spaces.

PROPOSITION 6.2. (see Theorem 3 of [8]). *If $(X, Y) \in \mathscr{L}\mathscr{S}(\sigma)$, then (X, Y) is strong intermediate for σ.*

We still need a further result from the work of Lorentz and Shimogaki:

PROPOSITION 6.3. (see Proposition 2 of [8]). *Let f be a positive decreasing function on $(0, l)$ and let $f = f_1 + f_2$, where $f_i \geqslant 0$, $i = 1, 2$. Then there exist positive decreasing functions f_i' with the properties $f_i' \prec f_i$, $i = 1, 2$, and $f_1' + f_2' = f$.*

LEMMA 6.4. *Suppose* $\bar{\sigma} = [(\Lambda(X_1), M^*(Y_1)), (\Lambda(X_2), M^*(Y_2))]$, *then we have*

(i) $g \prec f$ *with respect to* $\bar{\sigma}$ *implies* $g^*(t) \leqslant 2\, S(\sigma)[f^*](t)$, $t \in I$.

(ii) $g^*(t) \leqslant S(\sigma)[f^*](t)$, $t \in I$ *implies* $g \prec f$.

Proof. To prove (i), suppose $g \prec f$. By Lemma 4.3 there exist positive decreasing functions f_1 and f_2 such that $f_1 + f_2 = f^*$ and (4.9) holds. Since $g \prec f$ there exist functions g_1 and g_2 such that $g_1 + g_2 = g$, and

$$\sup_s g^*(s)\, \phi_{Y_i}(s) \leqslant \| f_i \|_{\Lambda(X_i)} \qquad i = 1, 2. \tag{6.1}$$

But, ϕ_{Y_i} is concave, so $\phi_{Y_i}(t)/2 \leqslant \phi_{Y_i}(t/2)$. By the inequality of Calderón, property (6.1), and (4.9) we obtain

$$\begin{aligned} g^*(t) &\leqslant g_1^*(t/2) + g_2^*(t/2) \leqslant 2\{\| f_1 \|_{\Lambda(X_1)}/\phi_{Y_1}(t) + \| f_2 \|_{\Lambda(X_2)}/\phi_{Y_2}(t)\} \\ &= 2(S(\sigma)[f_1](t) + S(\sigma)[f_2](t)) = 2S(\sigma)[f^*](t). \end{aligned}$$

For the proof of statement (ii) we first notice that $g \prec f$ is equivalent to $g \prec f^*$ since the spaces are rearrangement invariant. Suppose $f^* = f_1 + f_2$, then we can assume that $0 \leqslant f_i \leqslant f^*$ since $\Lambda(X_i)$ possesses property (1.1). By Proposition 6.3 we can find positive decreasing functions f_i' such that $f_1' + f_2' = f^*$ and $f_i' \prec f_i$, so

$$\| f_i' \|_{\Lambda(X_i)} \leqslant \| f_i \|_{\Lambda(X_i)} \,. \tag{6.2}$$

Since $g^*(t) \leqslant S(\sigma)[f_1'](t) + S(\sigma)[f_2'](t)$ and each of the functions $S(\sigma)[f_i']$ is positive and decreasing there are two positive decreasing functions g_1 and g_2 such that $g_1 + g_2 = g^*$ and $g_i(t) \leqslant S(\sigma)[f_i'](t)$. But $S(\sigma)$ is an operator of weak types (X_i, Y_i), so

$$g_i(t)\phi_{Y_i}(t) \leqslant S(\sigma)[f_i'](t)\, \phi_{Y_i}(t) \leqslant \| f_i \|_{\Lambda(X_i)} \,.$$

Taking the supremum over t and observing (6.2), we get statement (ii).

THEOREM 6.5. *Let* $\bar{\sigma}$ *be as above. The pair* (X, Y) *is strong intermediate for* $\bar{\sigma}$ *if and only if* (X, Y) *belongs to the class* $\mathscr{L}\mathscr{S}(\bar{\sigma})$.

Proof. The sufficiency follows from Proposition 6.2. For the necessity, suppose (X, Y) is strong intermediate for $\bar{\sigma}$. Then (X, Y) is weak intermediate for $\sigma = [(X_1, Y_1), (X_2, Y_2)]$. By Lemma 4.2, $S(\sigma)$ is a bounded operator from X to Y. Suppose $g \prec f$ with respect to $\bar{\sigma}$, then by Lemma 6.4

$$g^*(t) \leqslant 2S(\sigma)[f^*](t).$$

But this gives

$$\| g \|_Y \leqslant 2 \| S(\sigma) f^* \|_Y \leqslant 2 \| S(\sigma) \| \| f \|_X$$

and so $(X, Y) \in \mathscr{L}\mathscr{S}(\bar{\sigma})$.

Again, we let

$$\sigma = [(X_1, Y_1), (X_2, Y_2)] \quad \text{and} \quad \bar{\sigma} = [(\Lambda(X_1), M^*(Y_1)), (\Lambda(X_2), M^*(Y_2))].$$

We now state an interpolation method of Peetre and compare it with our results in Section 4.

DEFINITION 6.6. For a pair of Banach function spaces X_1, X_2 and a $t > 0$, define the K function norm on $X_1 + X_2$ by

$$K(t, f; X_1, X_2) = \inf_{f_1 + f_2 = f} \{ \| f_1 \|_{X_1} + t \| f_2 \|_{X_2} \}.$$

We denote by $\mathscr{P}(\sigma)$ the class of all pairs of Banach function spaces (X, Y) such that the following condition is satisfied:

For some constant C, $K(t, g; Y_1, Y_2) \leqslant K(t, f; X_1, X_2)$ all t implies $\| g \|_Y \leqslant C \| f \|_X$.

PROPOSITION 6.7. *If $(X, Y) \in \mathscr{P}(\sigma)$, then (X, Y) is strong intermediate for σ.*

This follows immediately from the elementary fact that

$$K(Tf, t; Y_1, Y_2) \leqslant (\max_{i=1,2} \| T \|_i) \, K(t, f; X_1, X_2).$$

This is true even if the Y_i are quasilinear.

The following lemma was essentially stated without proof in the introduction of [8].

LEMMA 6.8. *Suppose* (4.1) *holds, then*

$$K(t, f; \Lambda(X_1), \Lambda(X_2)) = \int_0^l f^*(s) \, d \min(\phi_{X_1}(s), t\phi_{X_2}(s)).$$

Proof. Since $\Lambda(X_1)$ and $\Lambda(X_2)$ are rearrangement invariant, we have $K(t, f^*) = K(t, f)$, so we may assume f is positive and decreasing. We prove that

$$K(t, f) = \inf_{\substack{f_1 + f_2 = f \\ f_i : \geqslant 0, \downarrow}} \{ \| f_1 \|_{\Lambda(X_1)} + t \| f_2 \|_{\Lambda(X_2)} \}. \tag{6.3}$$

Since $\Lambda(X_i)$ satisfy (1.1) it is certainly clear that we can assume $f_i \geqslant 0$. Using Proposition 6.3, we can select for each decomposition of f, say $f = f_1 + f_2$, two positive decreasing functions f_1', f_2' such that $f_1' + f_2' = f$ and $f_i' \prec f_i$. This gives that

$$\|f_i'\|_{\Lambda(X_i)} \leqslant \|f_i\|_{\Lambda(X_i)}, \qquad i = 1, 2,$$

and (6.3) follows.

So, now let $f = f_1 + f_2$, where f_i is positive and decreasing. Let δ be the interpolation segment $[(X_1, L^1), (X_2, L^\infty)]$. Note that $\phi_{L^1}(t) = t$, while $\phi_{L^\infty}(t) = 1$, if $t > 0$. In this case

$$t \cdot S(\delta)[f](t) = \int_0^l f(s) d \min(\phi_{X_1}(s), t\phi_{X_2}(s)).$$

Since $S(\delta)$ is of weak types (X_1, L^1) and (X_2, L^∞), we have

$$tS(\delta)[f_1](t) \leqslant \|f_1\|_{\Lambda(X_1)}$$

and

$$S(\delta)[f_2](t) \leqslant \|f_2\|_{\Lambda(X_2)}.$$

But

$$\begin{aligned}\int_0^l f(s) d \min(\phi_{X_1}(s), t\phi_{X_2}(s)) &= tS(\delta)[f](t) = tS(\delta)[f_1](t) + tS(\delta)[f_2](t)\\ &\leqslant \|f_1\|_{\Lambda(X_1)} + t\|f_2\|_{\Lambda(X_2)}.\end{aligned}$$

Taking the infinum over all such decompositions, we get

$$\int_0^l f(s) d \min(\phi_{X_1}(s), t\phi_{X_2}(s)) \leqslant K(t, f; \Lambda(X_1), \Lambda(X_2)).$$

For the reverse inequality we use Lemma 4.3 to get a decomposition of f, say $f_1 + f_2 = f$, where f_i is positive, decreasing and satisfies

$$S(\delta)[f_i](t) = \|f_i\|_{\Lambda(X_i)}\, t^{i-2} \qquad i = 1, 2.$$

Hence

$$\begin{aligned}K(t, f; \Lambda(X_1), \Lambda(X_2)) &\leqslant \|f_1\|_{\Lambda(X_1)} + t\|f_2\|_{\Lambda(X_2)}\\ &= t(S(\delta)[f_1](t) + S(\delta)[f_2](t)) = tS(\delta)[f](t)\\ &= \int_0^l f(s) d \min(\phi_{X_1}(s), t\phi_{X_2}(s)).\end{aligned}$$

Property (6.5) below provides insight to the selection of the operator $S(\sigma)$ as the choice for a maximal weak-type (X_i, Y_i) operator.

THEOREM 6.9. *Let $\bar{\sigma}$ be as above. The pair (X, Y) is strong intermediate for $\bar{\sigma}$ if and only if (X, Y) belongs to the class $\mathscr{P}(\bar{\sigma})$.*

Proof. The sufficiency is Proposition 6.7. To prove the necessity, notice that (X, Y) being strong intermediate for $\bar{\sigma}$ is equivalent to (X, Y) being weak intermediate for $\sigma = [(X_1, Y_1), (X_2, Y_2)]$. Now suppose

$$K(t, g; M^*(Y_1), M^*(Y_2)) \leqslant K(t, f; \Lambda(X_1), \Lambda(X_2)), \qquad \text{all } t. \tag{6.4}$$

For some constant C, we need $\| g \|_Y \leqslant C \| f \|_X$. But ϕ_{Y_i} is concave, so $(1/2) \cdot \phi_{Y_i}(t) \leqslant \phi_{Y_i}(t/2)$. Hence by the inequality of Calderón, (6.4), and Lemma 6.8 we have

$$\begin{aligned} g^*(t) &\leqslant 2/\phi_{Y_1}(t) \inf_{g_1+g_2=g} \left(\sup_s (g_1^*(s)\, \phi_{Y_1}(s)) + \frac{\phi_{Y_1}(t)}{\phi_{Y_2}(t)} \sup_s (g^*(s)\, \phi_{Y_2}(s))\right) \\ &= 2/\phi_{Y_1}(t)\, K\left(\frac{\phi_{Y_1}(t)}{\phi_{Y_2}(t)}, g; M^*(Y_1), M^*(Y_2)\right) \\ &\leqslant 2/\phi_{Y_1}(t)\, K\left(\frac{\phi_{Y_1}(t)}{\phi_{Y_2}(t)}, f; \Lambda(X_1), \Lambda(X_2)\right) \\ &= 2/\phi_{Y_1}(t) \cdot \int_0^l f(s) d \min\left(\phi_{X_1}(s), \frac{\phi_{Y_1}(t)}{\phi_{Y_2}(t)} \phi_{X_2}(s)\right) \\ &= 2S(\sigma)[f](t). \end{aligned} \tag{6.5}$$

But $S(\sigma)$ is a bounded operator from X to Y, so

$$\| g \|_Y \leqslant 2 \| S(\sigma) \| \| f \|_X .$$

Now we outline the definition and a few of the properties of the theory of scales of Banach spaces [5]. A family of Banach spaces $X_\alpha (0 \leqslant \alpha \leqslant 1)$ with norms $\| f \|_\alpha (f \in X_\alpha)$ is called a scale if

X_β is densely embedded in X_α when $\beta > \alpha$ and $\| f \|_\alpha \leqslant C(\alpha, \beta) \| f \|_\beta$. (6.6)

if $0 \leqslant \alpha < \beta < \gamma \leqslant 1$ there is a finite constant $C(\alpha, \beta, \gamma)$ such that

$$\| f \|_\beta \leqslant C(\alpha, \beta, \gamma) \| f \|_\alpha^{(\gamma-\beta)/(\gamma-\alpha)} \| f \|_\gamma^{(\beta-\alpha)/(\gamma-\alpha)} \qquad \text{for all } f \in X_\gamma . \tag{6.7}$$

By Corollary 3.3 and Proposition 3.4 it is not hard to see that $\{\Lambda_\alpha(X)\}_{0 \leqslant \alpha \leqslant 1}$ forms a scale, where $\Lambda_0(X)$ is set equal to $M^0(X)$. In fact, for this scale

$$\lim_{\alpha \to \beta} \| f \|_\beta = \| f \|_\alpha . \tag{6.8}$$

Hence the scale $\{\Lambda_\alpha(X)\}_{0\leqslant\alpha\leqslant 1}$ is continuous in the terminology of [5]. This scale can also be equivalently renormed so that $C(\alpha, \beta, \gamma) \leqslant 1$ and $C(\alpha, \beta) \leqslant e^e$. By Theorem 3.10 and Remark 3.11 we can see that the dual family $\{\Lambda_\alpha(X)^{*-}\}_{0\leqslant\alpha\leqslant 1}$ forms a scale where $\Lambda_\alpha(X)^{*-}$ is the closure of $L^1 \cap L^\infty$ in $\Lambda_\alpha(X)^*$. Hence the scale $\{\Lambda_\alpha(X)\}_{0\leqslant\alpha\leqslant 1}$ forms what is called a regular scale.

We define the rearrangement-invariant space $F_\alpha(X)$ as the closure of the set of all f such that

$$\|f\|_{F_\alpha(X)} = \sup_{\substack{g\neq 0\\ g\in\Lambda(X')}} \left(\int_0^l f^*(s)\, g^*(s)\, ds\right)\Big/\left(\|g\|_{\Lambda(X')}^{1-\alpha}\, \|g\|_{M(X')}^{\alpha}\right) < +\infty$$

in the $\|\ \|_{F_\alpha(X)}$ norm topology in $M^0(X)$. This construction is called the "minimal scale" in [5]. Theorem (4.2) of [5] gives that $\Lambda_\alpha(X)$ is continuously embedded in $F_\alpha(X)$ for each α. Also we have the following:

Proposition 6.10 (see Theorem (4.1) of [5]). *Each operator which maps $\Lambda(X)$ boundedly to $\Lambda(Y)$ and $M^0(X)$ boundedly to $M^0(Y)$, is also a bounded map from $\Lambda_\alpha(X)$ to $F_\alpha(X)$.*

Theorem (8.8) of [5] gives that $F_\beta(X)$ is continuously embedded in $\Lambda_\alpha(X)$ for $0 \leqslant \alpha < \beta \leqslant 1$. This together with Proposition 6.10 gives

Proposition 6.11 (see Theorem (8.9) of [5]). *Each operator which maps $\Lambda(X)$ boundedly to $\Lambda(Y)$ and $M^0(X)$ boundedly to $M^0(Y)$ is also a bounded map from $\Lambda_\beta(X)$ to $\Lambda_\alpha(Y)$, where $\alpha < \beta$.*

Theorem 6.12 (cf. Theorem 5.6) *Suppose* (5.5), (5.6), *and*

$$\lim_{t\to 0} F(s, t) = 0 \qquad s \in I \tag{6.9}$$

hold, then $(\Lambda_\beta(X), \Lambda_\alpha(Y))$ is weak intermediate for σ for each $0 \leqslant \alpha < \beta \leqslant 1$.

Proof. Condition (5.5) gives that $S(\sigma)$ maps $\Lambda(X)$ boundedly to $\Lambda(Y)$. Condition (5.6) implies $S(\sigma)$ maps $M(X)$ boundedly to $M(Y)$. This together with (6.9) implies $S(\sigma)$ maps $M^0(X)$ boundedly to $M^0(Y)$. We now apply Proposition 6.11 and Theorem 4.7.

Remark 6.13. Notice that Calderón's theorem involving L^{pq} spaces (and hence our Theorem 5.6) gives the Stein–Weiss theorem by the embedding of the L^p spaces between L^{pq} spaces. Theorem

6.12, however, only gives the Stein–Weiss theorem "off the diagonal," i.e., for $\sigma = [(L^{p_1}, L^{q_1}), (L^{p_2}, L^{q_2})]$, where either $p_1 \neq q_1$ or $p_2 \neq q_2$. In this sense we get better results than Krein and Petunin for our scales $\{\Lambda_\alpha(X)\}_{0 \leqslant \alpha \leqslant 1}$. On the other hand, Proposition 6.10 gives an interpolation result which we do not consider.

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References

1. D. W. Boyd, Indices of function spaces and their relationship to interpolation, *Canad. J. Math.* **21** (1969), 1245–1254.
2. P. L. Butzer and H. Berens, "Semigroups of Operators and Approximation," pp. 165–191, Springer-Verlag, New York, 1967.
3. A. P. Calderón, Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz, *Studia Math.* **26** (1966), 273–299.
4. G. H. Hardy, J. E. Littlewood, and G. Polya, "Inequalities," Cambridge University Press, Cambridge, 1934.
5. S. G. Krein and Yu. I. Petunin, Scales of Banach spaces, *Russian Math. Surveys* **21** (1966), 85–159.
6. G. G. Lorentz, "Bernstein Polynomials," University of Toronto Press, Toronto, 1953.
7. G. G. Lorentz, On the theory of spaces Λ, *Pacific J. Math.* **1** (1951), 411–429.
8. G. G. Lorentz and T. Shimogaki, Interpolation theorems for operators in function spaces, *J. Functional Analysis* **2** (1968), 31–51.
9. G. G. Lorentz and T. Shimogaki, Interpolation theorems for spaces Λ, *in* "Abstract Spaces and Approximation" (Proc. Conf., Oberwolfach, 1968), pp. 94–98, Birkhauser, Basel, 1969.
10. W. A. J. Luxemburg, Banach function spaces, Thesis Delft Institute of Technology, Assen, Netherlands, 1955.
11. E. T. Oaklander, "Interpolation, Espacios de Lorentz y Teorema de Marcinkiewicz," Cursos y seminarios Fasc. 20, Universidad de Buenos Aires, 1965.
12. E. M. Semenov, Imbedding theorems for Banach spaces of measurable functions, *Soviet Math. Dokl.* **5** (1964), 831–834.
13. E. M. Semenov, A new interpolation theorem, *Functional Anal. Appl.* **2** (1968), 158–168.
14. T. Shimogaki, A note on norms of compression operators on function spaces, *Proc. Japan Acad.* **46** (1970), 239–242.
15. T. Shimogaki, Indices of function spaces, preprint.
16. M. Zippin, Interpolation of operators of weak type between rearrangement invariant function spaces, *J. Functional Analysis* **7** (1971), 267–284.