NOTE ON VECTOR POTENTIALS

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Assume \( \mathbf{F} \) is a vector field on \( \mathbb{R}^3 \) (for the argument below all we need is a star shaped domain) with continuous second order partials. Then \( \mathbf{F} = \nabla \times \mathbf{G} \) implies that \( \nabla \cdot \mathbf{F} = 0. \) Conversely if \( \nabla \cdot \mathbf{F} = 0, \) then \( \mathbf{G} \) given by

\[
\mathbf{G}(x, y, z) = \int_0^1 \mathbf{F}(xt, yt, zt) \times (txi + tyj + tzk) \, dt
\]

is a vector potential of \( \mathbf{F}, \) i.e., \( \mathbf{F} = \nabla \times \mathbf{G}. \) The usual demonstration of this uses various vector identities. Here we present a straightforward calculation to show this. We will only show that the \( i \)-component of \( \nabla \times \mathbf{G} \) equals \( \mathbf{F}_1, \) where we will write \( \mathbf{F} = F_1i + F_2j + F_3k. \) Note first that \( \nabla \cdot \mathbf{F} = 0 \) implies that \( \frac{\partial F_1}{\partial x} = -\frac{\partial F_2}{\partial y} - \frac{\partial F_3}{\partial z}. \) To shorten the notation we denote by \( F_t^i \) the scalar field \( F_t^i(x, y, z) = F_i(x t, y t, z t) \) \( (i = 1, 2, 3). \) Now the integrand (of the integral defining \( \mathbf{G} \)) is given by

\[
(ztF_2 - ytF_3)i - (ztF_2 - xtF_3)j + (ytF_1 - xtF_2)k
\]

Now the \( i \)-component of the curl of this vector field equals

\[
\frac{\partial}{\partial y}(ytF_1^t - xtF_2^t) - \frac{\partial}{\partial z}(xtF_3^t - ztF_1^t) =
\]

\[
tF_1^t + yt^2 \frac{\partial F_1^t}{\partial y} - xt^2 \frac{\partial F_2^t}{\partial y} - xt^2 \frac{\partial F_3^t}{\partial z} + tF_1^t + zt^2 \frac{\partial F_1^t}{\partial z} =
\]

\[
2tF_1^t + xt^2 \frac{\partial F_1^t}{\partial x} + yt^2 \frac{\partial F_1^t}{\partial y} + zt^2 \frac{\partial F_1^t}{\partial z} =
\]

\[
\frac{d}{dt}(t^2F_1^t).
\]

Hence the \( i \)-component of the curl of \( \mathbf{G} \) is given by

\[
\int_0^1 \frac{d}{dt}(t^2F_1^t) \, dt = (t^2F_1^t)|_{t=1}^{t=0} = F_1.
\]

**Homework problems**

In the following three problems determine a vector potential for the given vector field or explain why no such potential exists.

1. \( \mathbf{F} = 2xi - yj - zk. \)
2. \( \mathbf{F} = xi - yj + zk. \)
3. \( \mathbf{F} = 3yi + 2xzj - 4x^2yk. \)