Solutions for HW 8

Pg. 93: 15 Solution: Note first that \( \int f(x) \, dx = \left[ 2x^2 \right]_0^1 = 2 \). By translation invariance this implies that also \( \int f(x - r_n) \, dx = 2 \). This implies by a corollary to the MCT that

\[
\int F(x) \, dx = \sum_{n=1}^{\infty} \int 2^{-n} f(x - r_n) \, dx = 2.
\]

Hence \( F \) is integrable and the series converges a.e. to \( F \). Assume \( \tilde{F} = F \) a.e. and let \( I \) be non-empty open interval. Then

\[
\int \tilde{F}(x) \, dx = \sum_{n=1}^{\infty} \int 2^{-n} f(x - r_n) \, dx = 2
\]
a.e. Let \( r_N \in I \) and \( I_0 \) the set of \( x \in I \) for which \( \tilde{F} \) equals the series. Then for any \( \epsilon > 0 \) there exists \( x \in I_0 \) with \( |x - r_N| < \epsilon \). Then \( \tilde{F}(x) \geq 2^{-N} \epsilon^{-\frac{1}{2}} \), which by letting \( \epsilon \to 0 \) shows that \( \tilde{F} \) is unbounded on \( I \).

Extra Problem Solution We can assume that \( f \) is not \( 0 \) a.e.

1. From \( ||f||^p \leq ||f||^r ||f||^{r-p} \) a.e., we get \( \int ||f||^p \, dx \leq (\int ||f||^r \, dx)(||f||^{r-p}) \) for all \( p > r \), which is the required inequality if we take \( p \)-th-roots on both sides of the inequality.

2. Since \( 0 < ||f||_r < \infty \) it follows that \( \lim_{p \to \infty} ||f||^{r/p} = 1 \). Hence from (1) we get \( \lim ||f||_p \leq ||f||_{\infty} \). Now let \( 0 < t < ||f||_{\infty} \). Then put \( A = \{ x : ||f(x)|| \geq t \} \), so \( m(A) > 0 \). Then \( t^n m(A) \leq \int_A ||f(x)||^p \, dx \leq \int ||f||^p \, dx \) implies (by taking \( p = r \)) that \( m(A) < \infty \). Now \( t(m(A)^{1/p}) \leq ||f||_p \) implies that \( t \leq \lim_{p \to \infty} ||f||_p \). As this holds for all \( t < ||f||_{\infty} \), we conclude that \( ||f||_{\infty} \leq \lim ||f||_p \). Hence \( ||f||_{\infty} = \lim ||f||_p \).

3. Solution: Assume first that \( (f_n) \) converges to \( f \) in \( L_p \). Then \( ||f||_p \leq ||f - f_n||_p + ||f_n||_p \) implies that \( ||f||_p - ||f_n||_p \leq ||f - f_n||_p \). Similarly \( ||f_n||_p - ||f||_p \leq ||f - f_n||_p \) and thus \( ||f||_p - ||f_n||_p \leq ||f - f_n||_p \), which shows that \( \lim ||f_n||_p = ||f||_p \). Note in this direction we do not need that \( f_n(x) \to f(x) \) a.e. For the converse observe that \( ||f - f_n||_p \leq 2^p (||f||^p + ||f_n||^p) \). Now put \( g_n = 2^p (||f||^p + ||f_n||^p) - ||f - f_n||^p \). Then \( \lim g_n(x) = 2^{p+1} ||f||^p \) a.e. By Fatou’s Lemma we get

\[
2^{p+1} \int ||f||^p \, dx = \int \lim g_n \, dx \leq \lim \inf \int g_n \, dx
\]

\[
= 2^{p+1} \int ||f||^p \, dx - \lim \sup \int ||f - f_n||^p \, dx.
\]

Hence we conclude that \( \int ||f - f_n||^p \, dx \to 0 \). \( ||f - f_n||_p \to 0 \).

(3) If \( g \in L^1 \) is continuous with compact support, the \( ||g||_1 = ||g||_1 \) and for all \( x \) we have \( g_n(x) \to g(x) \) as \( h \to 0 \). By a previous hw this implies that \( ||g_h - g||_1 \to 0 \) as \( h \to 0 \) (note this can also be proved directly using uniform
continuity). Let $\epsilon > 0$. Then there exists $g$ continuous with compact support such that $\|f - g\|_1 < \frac{\epsilon}{3}$. This implies by translation invariance that also $\|f_h - g_h\|_1 < \frac{\epsilon}{3}$ for any $h$. Now there exists $h_0 > 0$ such that $\|g - g_h\|_1 < \frac{\epsilon}{3}$ for all $|h| < h_0$. Then we have for all $|h| < h_0$ that

$$\|f - f_h\|_1 \leq \|f - g\|_1 + \|g - g_h\|_1 + \|g_h - f_h\|_1 < \epsilon.$$