Solutions for HW 6

Pg. 91: 6 Solution:

(1) We define $f$ piecewise. Let $\epsilon_n = \frac{1}{2^n}$. Define first $f$ on $[2, \infty)$ as follows. For $n \geq 2$ let $f(n) = f(n + \frac{1}{2^n}) = \epsilon_n$, $f(n + \frac{1}{2^{n+1}}) = n$, and $f$ linear and continuous on $[n, n + \frac{1}{2^n}]$, $[n + \frac{1}{2^n}, n + \frac{1}{2^{n+1}}]$, and $[n + \frac{1}{2^{n+1}}, n + 1]$. Then $f$ positive and continuous on $[2, \infty)$ and $\int_n^{n+1} f \, dx \leq \frac{1}{2^n} + \epsilon_n$ implies that $\int_2^\infty f \, dx < \infty$. Moreover $\lim_{x \to \infty} f(x) = \infty$. To get a function on $\mathbb{R}$ extend the above $f$ by defining $f(x) = e^{2e^{-2}}$ on $(-\infty, 2]$.

(2) Assume $\lim_{x \to -\infty} |f(x)| = \epsilon > 0$. Then by uniform continuity there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{4}$. Then there exist $x_n \uparrow, x_n \to \infty$ such that $x_{n+1} - x_n > 2\delta$ and $|f(x_n)| > \frac{\epsilon}{2}$. This implies that $|f(y)| > \frac{\epsilon}{4}$ for all $y \in (x_n - \delta, x_n + \delta)$. Hence $\int_{x_n - \delta}^{x_n + \delta} |f| \, dx \geq \frac{\epsilon}{2}$ for all $n$. But $x_{n+1} - x_n > 2\delta$ implies $x_n + \delta < x_{n+1} - \delta$ so that the intervals $(x_n - \delta, x_n + \delta)$ are disjoint. Hence $\int |f| \, dx \geq \sum_n \int_{x_n - \delta}^{x_n + \delta} |f| \, dx = \infty$. This contradiction with the integrability of $f$ shows that $\lim_{x \to -\infty} f(x) = 0$.

Pg. 91: 10 Solution From $2^k \chi_{F_k} \leq f \chi_{F_k} \leq 2^{k+1} \chi_{F_k}$ we get that

$$2^k m(F_k) \leq \int_{F_k} f \, dx \leq 2^{k+1} m(F_k).$$

This implies that

$$\sum_{k = -\infty}^\infty 2^k m(F_k) \leq \int_{\mathbb{R}} f \, dx \leq 2 \sum_{k = -\infty}^\infty 2^k m(F_k),$$

which proves the first equivalence. Since $E_{2^n} = \cup_{k=n}^\infty F_k$ is a disjoint union, we have $m(E_{2^n}) = \sum_{k=n}^\infty m(F_k)$. This implies that

$$\sum_{n = -\infty}^\infty 2^n m(E_{2^n}) = \sum_{n = -\infty}^\infty 2^n m(F_k) = \sum_{k = -\infty}^\infty m(F_k) \sum_{n = -\infty}^k 2^n = 2 \sum_{k = -\infty}^\infty 2^k m(F_k),$$

which implies the remaining equivalence. An alternate way is to use that $m(E_{2^k}) = m(F_k) + m(E_{2^{k+1}})$. Assuming that $f$ is integrable, this implies that $m(F_k) = m(E_{2^k}) - m(E_{2^{k+1}})$. Hence

$$2 \sum_{k = -\infty}^\infty 2^k m(F_k) = 2 \sum_{k = -\infty}^\infty 2^k m(E_{2^k}) - \sum_{k = -\infty}^\infty 2^{k+1} m(E_{2^{k+1}})$$

$$= \sum_{k = -\infty}^\infty 2^k m(E_{2^k}).$$
Let now \( f(x) = |x|^{-\alpha} \chi_{B(0,1)} \). Then \( E_{2k} = B(0;1) \) for all \( k \leq -1 \) and \( E_{2k} = B(0,\frac{1}{2^k}) \) for \( k \geq 0 \). Now \( m\left(B(0,\frac{1}{2^k})\right) = \frac{1}{2^k} m(B(0,1)) \). This implies that

\[
\sum_{k=-\infty}^{\infty} 2^k m(E_{2k}) < \infty
\]

if and only if

\[
\sum_{k=0}^{\infty} 2^{k-\frac{k_d}{2}} < \infty,
\]

which is if and only if \( d > a \). Now let \( f = |x|^{-b} \chi_{\{x:|x|\geq 1\}} \). We can assume \( b > 0 \). Then \( E_{2k} = \emptyset \) for \( k \geq 0 \) and \( E_{2k} = \{x : 1 \leq |x| < 2^{-\frac{k}{2}}\} \). Now

\[
\sum_{k=-\infty}^{\infty} 2^k m(E_{2k}) < \infty
\]

if and only if

\[
\sum_{k=-\infty}^{\infty} 2^{k-\frac{k_d}{2}} m(B(0,1)) < \infty.
\]

This holds if and only if \( \sum_{k=-\infty}^{\infty} 2^{k-\frac{k_d}{2}} m(B(0,1)) < \infty \), which holds if and only if \( b > d \).

**Pg. 91:11 Solution** Let \( E = \{x : f(x) < 0\} \). Then \( \int_E f \, dx \leq 0 \), and thus \( \int_E f \, dx = 0 \) from the hypothesis. This implies that \( f \chi_E = 0 \) a.e., which proves \( m(E) = 0 \). Hence \( f \geq 0 \) a.e.

**Extra Problem**

(1) Proof: Let \( \epsilon > 0 \). Then there exists \( N_1 \) such that \( a - \epsilon < a_n < a + \epsilon \) for all \( n \geq N_1 \). If \( \lim b_n = b \) is finite, then there exists \( N_2 \) such that \( b - \epsilon < b_n \) for all \( n \geq N_2 \) and \( b_n < b + \epsilon \) for infinitely many \( n \). This shows \( a + b - 2\epsilon < a_n + b_n \) for all \( n \geq \max\{N_1, N_2\} \) and \( a_n + b_n < a + b + 2\epsilon \) for infinitely many \( n \). This shows \( \lim a_n + b_n = a + b \). If \( b = -\infty \), then for all \( M \) there exists infinitely many \( n \) such that \( b_n < M \). Hence \( a_n + b_n < M + a + \epsilon \) for infinitely many \( n \). This shows that in this case \( \lim a_n + b_n = -\infty = b + a \). In case \( b = \infty \), then for all \( M \) there exists \( N_2 \) such that \( b_n > M \) for all \( n \geq N_2 \). This implies that \( a_n + b_n > M + a - \epsilon \) for all \( n \geq \max\{N_1, N_2\} \), which shows \( \lim a_n + b_n = \infty = a + b \).

(2) Let \( g_n = |f| + |f_n| - |f - f_n| \). Then \( 0 \leq g_n \) is measurable and \( g_n(x) \to 2|f(x)| \) a.e. By Fatou’s Lemma

\[
\int 2|f| \, dx \leq \lim \int g_n \, dx = \lim \int |f| + |f_n| - |f - f_n| \, dx.
\]

By part (1) we have

\[
\lim \int |f| + |f_n| - |f - f_n| \, dx = 2 \int |f| \, dx + \lim \int |f - f_n| \, dx
\]

\[
= 2 \int |f| \, dx - \lim \int |f - f_n| \, dx.
\]

Combining this with the above inequality we \( \lim \int |f - f_n| \, dx \leq 0 \), which implies \( \lim \int |f - f_n| \, dx = 0 \).