Solutions for HW 5

Pg. 312: 3 Solution:
Assume first that $E$ is Lebesgue measurable and let $A$ be given. Then there exists a measurable set $G \supset A$ such that $m^*(A) = m(G)$. Then $m^*(A) = m(G) = m(G \cap E) + m(G \cap E^c) \geq m^*(A \cap E) + m^*(A \cap E^c)$. By subadditivity we always have have $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$, so $E$ is Caratheodory measurable.

Assume now that $E$ is Caratheodory measurable. Let $E_n = E \cap B(0;n)$. Then there exist for each $n$ a measurable set $G_n \supset E_n$ such that $m(G_n) = m^*(E_n)$. We can assume that $G_n \subset B(0;n)$. Then $G_n \cap E = G_n \cap E \cap B(0;n) = G_n \cap E_n = E_n$.

Taking now $A = G_n$ in the Caratheodory measurability property of $E$ we find $m(G_n) = m^*(G_n \cap E) + m^*(G_n \setminus E) = m^*(E_n) + m^*(G_n \setminus E)$. Hence $m^*(G_n \setminus E) = 0$, as $m(G_n) < \infty$. Let $G = \cup_n G_n$. Then $G$ measurable, $G \supset E$, and $m^*(G \setminus E) = 0$.

Hence $G \setminus E$ is measurable and thus $E = G \setminus (G \setminus E)$ is measurable.

Pg. 91:9 Solution From $\alpha \chi_{E_n} \leq f$ it follows that $\alpha m(E_n) \leq \int f \, dx$. 

1