Solutions homework 11.

(1) Let \( F, G : [a, b] \to \mathbb{R} \) be absolutely continuous functions. Prove that the product \( FG \) is absolutely continuous.

**Solution:** Since \( F, G \) are continuous, they are bounded on \([a, b]\), say \( |F(x)| \leq M \) and \( |G(x)| \leq N \) for all \( x \in [a, b] \). Let \( \epsilon > 0 \). Then there exists \( \delta_1 > 0 \) such that if \( \{(a_i, b_i)\} \) is finite disjoint collection of open intervals in \([a, b]\) with \( \sum (b_i - a_i) < \delta_1 \), then

\[
\sum |F(b_i) - F(a_i)| < \frac{\epsilon}{2M}.
\]

Similarly here exists \( \delta_2 > 0 \) such that if \( \{(a_i, b_i)\} \) is finite disjoint collection of open intervals in \([a, b]\) with \( \sum (b_i - a_i) < \delta_2 \), then

\[
\sum |G(b_i) - G(a_i)| < \frac{\epsilon}{2N}.
\]

Let \( \delta = \min\{\delta_1, \delta_2\} \). If now \( \{(a_i, b_i)\} \) is finite disjoint collection of open intervals in \([a, b]\) with \( \sum (b_i - a_i) < \delta \), then

\[
\sum |F(b_i)G(b_i) - F(a_i)G(a_i)| \leq \sum |F(b_i)G(b_i) - F(b_i)G(a_i)| + |F(b_i)G(a_i) - F(a_i)G(a_i)|
\]

\[
< M \sum |G(b_i) - G(a_i)| + N \sum |F(b_i) - F(a_i)| < \epsilon.
\]

Hence \( FG \) is absolutely continuous.

(2) Let \( F : [0, 1] \to \mathbb{R} \) such that \( F'(x) \) exists a.e. and satisfies \( F' \in L^1([0, 1]) \). Assume \( F \) is continuous at 0 and absolutely continuous on \([\epsilon, 1]\) for all \( \epsilon > 0 \). Prove that \( F \) is absolutely continuous on \([0, 1]\) and thus of bounded variation on \([0, 1]\).

**Solution:** Let \( 0 < x \leq 1 \) and \( 0 < \epsilon < x \). Then

\[
F(x) = F(\epsilon) + \int_{\epsilon}^{x} F'(y) \, dy.
\]

Now \( F(\epsilon) \to F(0) \) as \( \epsilon \to 0 \) by the continuity of \( F \) at 0. For any sequence \( \epsilon_n \to 0 \) the sequence of functions \( F'(y) \chi_{[\epsilon_n, x)}(y) \) converges \( F'(y) \chi_{[0, x]}(y) \) a.e. and the sequence is bounded above by \( |F'| \in L^1[0, 1] \). Hence by the Dominated Convergence Theorem

\[
\int_{\epsilon_n}^{x} F'(y) \, dy \to \int_{0}^{x} F'(y) \, dy.
\]

This implies that

\[
F(x) = F(0) + \int_{0}^{x} F'(y) \, dy
\]

and thus \( F \) is absolutely continuous.

(3) Let \( a > b > 0 \) and define \( F(0) = 0, \ F(x) = x^a \sin \frac{1}{x^b} \) for \( 0 < x \leq 1 \). Prove that \( F \) is of bounded variation on \([0, 1]\). **Solution:** As \( a > 0 \) we have that \( |F'(x)| \leq x^{a-1} \to 0 \) as \( x \to 0 \), so \( F \) is continuous at 0. Also for \( x \neq 0 \) we have \( F'(x) = ax^{a-1} \sin \frac{1}{x^b} - bx^{a-b-1} \cos \frac{1}{x^b} \). Hence \( |F'(x)| \leq ax^{a-1} + bx^{a-b-1} \in L^1[0, 1] \), so \( F' \in L^1[0, 1] \). For \( \epsilon > 0 \) the function \( F' \) is continuous on \([\epsilon, 1]\), and thus bounded. This implies that \( F \) is Lipschitz on \([\epsilon, 1]\) and thus absolutely continuous on \([\epsilon, 1]\). From problem 2 it follows that \( F \) is absolutely continuous and thus of bounded variation on \([0, 1]\).