MATH 555/704I NOTES
LEBESGUE'S THEOREM ON RIEMANN INTEGRABLE FUNCTIONS

In this note we will present a self contained proof of Lebesgue's characterization of Riemann integrable functions on $[a, b]$ as those bounded functions which are continuous a.e. on $[a, b]$. The text contains a proof which uses the theory of the Lebesgue integral. Other proofs exist which use either the notions of semi-continuous functions or the modulus of continuity of a function. Our proof is modelled after one which uses the theory of the Lebesgue integral, but only uses the concepts of measure (and content) zero. Recall that $E \subset \mathbb{R}$ has (Lebesgue) *measure zero* if for all $\epsilon > 0$ there exists a countable collection $\{I_n\}$ of open intervals with $E \subset \cup_{n=1}^{\infty} I_n$ such that $\sum_n l(I_n) < \epsilon$, where $l(I_n)$ denotes the length of the interval I_n. If we replace in this definition the countable collection of intervals by a *finite* collection of of open intervals, then we say that E has *content zero*. Obviously a set of content zero has measure zero.

Lemma 1. *A countable union of sets of measure zero has measure zero.*

Proof. Let $E_n \subset \mathbb{R}$ have measure zero and put $E = \cup_n E_n$. Let $\epsilon > 0$. Then for each n there exist a countable collection $\{I_{n,k}\}_{k=1}^{\infty}$ of open intervals such that $E_n \subset \cup_{k=1}^{\infty} I_{n,k}$ and $\sum_k l(I_{n,k}) < \frac{\epsilon}{2^n}$. Now $\{I_{n,k}\}_{k,n}$ is again a countable union of open intervals and $E \subset \cup_{k,n} I_{n,k}$ such that $\sum_{k,n} l(I_{n,k}) < \epsilon$. Hence E has measure zero. □

Lemma 2. *Let $0 \le f : [a, b] \to \mathbb{R}$ be a Riemann integrable function with $\int_a^b f = 0$. Then for all $c > 0$ the set $\{x \in [a, b] : f(x) \ge c\}$ has content zero.*

Proof. Let $c > 0$ and denote by E the set $\{x \in [a, b] : f(x) \ge c\}$. Let $\epsilon > 0$. Then there exists a partition $\mathcal{P} = \{x_0, \dots, x_n\}$ of $[a, b]$ such that $\mathcal{U}(\mathcal{P}, f) < \epsilon \cdot c$, where $\mathcal{U}(\mathcal{P}, f)$ denotes the Riemann upper sum corresponding to $\mathcal{P}$, i.e.,

$$\mathcal{U}(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \Delta x_i,$$

where

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.$$

Denote $I = \{i : E \cap [x_{i-1}, x_i] \ne \emptyset\}$. If $i \in I$, then $M_i \ge c$. Hence we have

$$\epsilon \cdot c > \mathcal{U}(\mathcal{P}, f) \ge \sum_{i \in I} M_i \Delta x_i \ge c \sum_{i \in I} \Delta x_i.$$

From this it follows that $\sum_{i \in I} l([x_{i-1}, x_i]) = \sum_{i \in I} \Delta x_i < \epsilon$. Since E is covered by $\{[x_{i-1}, x_i] : i \in I\}$, it follows that E has content zero. □

We say that a property P holds almost everywhere (abbreviated by a.e.) on $[a, b]$, if the set $\{x \in [a, b] : P \text{ fails for x}\}$ has measure zero.

Corollary 3. *Let $0 \le f : [a,b] \to \mathbb{R}$ be a Riemann integrable function with $\int_a^b f = 0$. Then f is zero a.e. on $[a,b]$.*

Proof. The set $\{x \in [a,b] : f(x) \neq 0\} = \cup_{n=1}^{\infty} \{x \in [a,b] : f(x) \geq \frac{1}{n}\}$, which is by the above lemma a countable union of sets of content zero and has thus measure zero. $\square$

Let $\mathcal{P} = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$. Let

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

and

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

Denote by ϕ the upper function for f corresponding to $\mathcal{P}$ and by ψ the corresponding lower function, i.e.,

$$\phi(x) = \sum_{i=1}^{n} M_i \chi_{[x_{i-1}, x_i)}(x)$$

and

$$\psi(x) = \sum_{i=1}^{n} m_i \chi_{[x_{i-1}, x_i)}(x).$$

It is easy to see that ϕ and ψ are Riemann integrable and that $\int_a^b \phi = \mathcal{U}(\mathcal{P}, f)$ and $\int_a^b \psi = \mathcal{L}(\mathcal{P}, f)$. Moreover $\psi(x) \leq f(x) \leq \phi(x)$ on $[a,b)$. Recall that the upper Riemann integral is given by $\overline{\int_a^b} f = \inf\{\mathcal{U}(\mathcal{P}, f) : \mathcal{P} \text{ partition of } [a,b]\}$ and that the lower Riemann integral is given by $\underline{\int_a^b} f = \sup\{\mathcal{L}(\mathcal{P}, f) : \mathcal{P} \text{ partition of } [a,b]\}$. By definition f is Riemann integrable if the lower integral of f equals the upper integral of f.

Theorem 4 (Lebesgue). *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous a.e. on $[a,b]$.*

Proof. Assume first that f is Riemann integrable on $[a,b]$. Let $\{\mathcal{P}_k\}$ be a sequence of partitions of $[a,b]$ with $\mathcal{P}_k \subset \mathcal{P}_{k+1}$ and such that the mesh $|\mathcal{P}_k| \to 0$ as $k \to \infty$. Let ϕ_k the upper function for f corresponding to $\mathcal{P}_k$ and by ψ_k the corresponding lower function. Then $\psi_k(x) \uparrow \leq f(x) \leq \phi_k(x) \downarrow$ for all $x \in [a,b)$ and $\int_a^b \psi_k \uparrow \int_a^b f$ and $\int_a^b \phi_k \downarrow \int_a^b f$. Let $g(x) = \lim_{k\to\infty} \psi_k(x)$ and $h(x) = \lim_{k\to\infty} \phi_k(x)$ for $x \in [a,b]$. It follows now that $\psi_k(x) \leq g(x) \leq f(x) \leq h(x) \leq \phi_k(x)$ for $x \in [a,b)$. Hence we have

$$\int_a^b \psi_k \leq \underline{\int_a^b} g \leq \overline{\int_a^b} g \leq \int_a^b f \leq \underline{\int_a^b} h \leq \overline{\int_a^b} h \leq \int_a^b \phi_k.$$

Letting $k \to \infty$ we conclude that g and h are Riemann integrable and that $\int_a^b g = \int_a^b h = \int_a^b f$. As $h \geq g$ it follows from Corollary 3 that $g = h$ a.e. Hence the set $E = \{x \in [a,b] : g(x) \neq h(x)\} \cup \bigcup_k \mathcal{P}_k$ has measure zero. We claim that f is continuous on $[a,b] \setminus E$. Let $x_0 \in [a,b] \setminus E$ and let $\epsilon > 0$. Then $g(x_0) = h(x_0)$ implies that there exists $k \in \mathbb{N}$ such that $\phi_k(x_0) - \psi_k(x_0) < \epsilon$. Now $\phi_k - \psi_k$ is constant in a neighborhood of x_0, since $x_0 \notin \mathcal{P}_k$. Hence there exists $\delta > 0$ such that $\phi_k(x) - \psi_k(x) = \phi_k(x_0) - \psi_k(x_0)$ for all $|x - x_0| < \delta$. For $|x - x_0| < \delta$ we now have

$$-\epsilon < \psi_k(x_0) - \phi_k(x_0) \leq f(x) - f(x_0) \leq \phi_k(x_0) - \psi_k(x_0) < \epsilon,$$

which shows that f is continuous at x_0. This completes the proof that f is continuous except for a set of measure zero. Assume now that f is continuous on $[a, b] \setminus E$, where E has measure zero. Let $\epsilon > 0$ and M such that $|f(x)| \leq M$ on $[a, b]$. Then $|f(x) - f(y)| \leq 2M$ for all $x, y \in [a, b]$. Since E has measure zero, there exists open intervals $I_1, I_2, \ldots$ such that $E \subset \cup_n I_n$ and $\sum_n l(I_n) < \frac{\epsilon}{4M}$. For all $x \in [a, b] \setminus E$ there exists an open interval J_x with $x \in J(x)$ such that $|f(z) - f(y)| \leq \frac{\epsilon}{2(b-a)}$ for all $y, z \in J_x \cap [a, b]$, since f is continuous at such x. Now $\{I_k\} \cup \{J_x : x \in [a, b] \setminus E\}$ is an open cover of $[a, b]$, so by compactness of $[a, b]$ there exists a finite cover $\{I_k\}_{k=1}^n \cup \bigcup_{i=1}^m \{J_{x_i} : x_i \in [a, b] \setminus E\}$ of $[a, b]$. Let $\mathcal{P} = \{a = t_0, \ldots, t_N = b\}$ be the partition of $[a, b]$ determined by those endpoints of $\{I_k\}_{k=1}^n$ and $\{J_{x_i} : x_i \in [a, b] \setminus E\}$, which are inside $[a, b]$. For each $1 \leq j \leq N$ the interval (t_{j-1}, t_j) is contained in some I_k or some J_{x_i}. Let $J = \{j : (t_{j-1}, t_j) \subset I_k \text{ for some } k\}$. Then we have that

$$\begin{aligned}
\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) &= \sum_{j=1}^{N} \Delta(t_j) \cdot \sup\{f(x) - f(y) : x, y \in [t_{j-1}, t_j]\} \\
&\leq \sum_{j \in J} \Delta(t_j) \cdot 2M + \sum_{j \notin J} \Delta(t_j) \cdot \frac{\epsilon}{2(b-a)} \\
&< \frac{\epsilon}{4M} \cdot 2M + (b-a) \cdot \frac{\epsilon}{2(b-a)} = \epsilon.
\end{aligned}$$

Hence f is Riemann integrable. □

Exercise 1. Prove that a set E has content zero if and only if there exists a closed bounded interval $[a, b]$, containing E, such that χ_E is Riemann integrable on $[a, b]$ and has Riemann integral zero.

Exercise 2. Prove that a set has zero content if and only if its closure is a bounded set with measure zero.

Exercise 3. Give an example of a bounded set with measure zero which does not have content zero.